Progress Report on Euler-Lagrange Equation and Passivity

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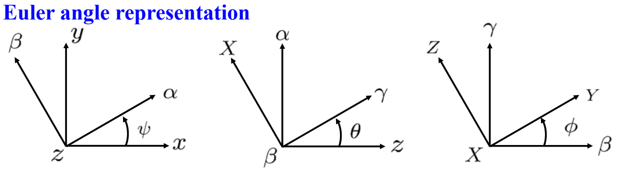
FL13-7-2

Angular velocity

Euler angle representation



xyz → Yaw rotation → $\alpha\beta z$ → Pitch rotation → $X\beta\gamma$ → Roll rotation → XYZ

$$\begin{bmatrix} \boldsymbol{E}_\alpha \\ \boldsymbol{E}_\beta \end{bmatrix} = \begin{bmatrix} C\psi & S\psi \\ -S\psi & C\psi \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_x \\ \boldsymbol{E}_y \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{E}_\gamma \\ \boldsymbol{E}_X \end{bmatrix} = \begin{bmatrix} C\theta & S\theta \\ -S\theta & C\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_z \\ \boldsymbol{E}_\alpha \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{E}_Y \\ \boldsymbol{E}_Z \end{bmatrix} = \begin{bmatrix} C\phi & S\phi \\ -S\phi & C\phi \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_\beta \\ \boldsymbol{E}_\gamma \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{\Omega} &= \dot\phi \boldsymbol{E}_X + \dot\theta \boldsymbol{E}_\beta + \dot\psi \boldsymbol{E}_z \\ &= (\dot\phi - \dot\psi \sin\theta)\boldsymbol{E}_X + (\dot\theta + \dot\psi \cos\theta \sin\theta)\boldsymbol{E}_Y \\ &\quad + (\dot\psi \cos\theta\cos\phi - \dot\theta \sin\phi)\boldsymbol{E}_Z \\ &= \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} \end{aligned}$$

Review

Euler Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right] - \frac{\partial L}{\partial \boldsymbol{q}} = \mathbf{Q}'$$

L :lagrangian
$\boldsymbol{q}$:generalized coordinate
$\mathbf{Q}'$:generalized force

⇩

$$M(\boldsymbol{\eta})\ddot{\boldsymbol{\eta}} + C(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}} = \mathbf{Q}'$$

Lagrangian

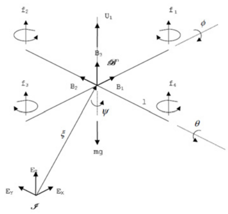
$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T - U$$

T :kinetic energy
U :potential energy

Review

Generalized coordinate

$$\boldsymbol{q} = (x, y, z, \phi, \theta, \psi)^T \in \mathbb{R}^6$$
$$\boldsymbol{\xi} = (x, y, z)^T \in \mathbb{R}^3$$
$$\boldsymbol{\eta} = (\phi, \theta, \psi)^T \in \mathbb{R}^3$$



Translational kinetic energy

$$T_{trans} = \frac{m}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}}$$

m:the mass of the quad-rotor

Rotational kinetic energy

$$T_{rot} = \frac{1}{2}\boldsymbol{\Omega}^T \boldsymbol{I} \boldsymbol{\Omega}$$

$\boldsymbol{\Omega}$:angular velocity $\quad I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$:inertia matrix

Potential energy

$$U = mgz$$

g:acceleration of gravity $\quad z$:rotorcraft altitude

⇨ $$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{m}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}} + \frac{1}{2}\boldsymbol{\Omega}^T \boldsymbol{I} \boldsymbol{\Omega} - mgz$$

Review

Angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{W}_\eta \dot{\boldsymbol{\eta}} \qquad \boldsymbol{W}_\eta = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \qquad \mathbb{J}(\boldsymbol{\eta}) = \boldsymbol{W}_\eta^T \boldsymbol{I} \boldsymbol{W}_\eta$$

$$T_{rot} = \frac{1}{2}\dot{\boldsymbol{\eta}}^T \mathbb{J} \dot{\boldsymbol{\eta}}$$

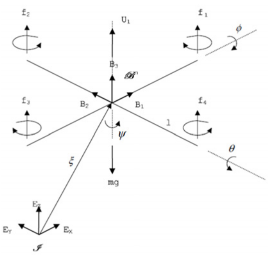
Generalized force

$$\mathbf{Q}' = \begin{bmatrix} \boldsymbol{F}_\xi \\ \boldsymbol{\tau} \end{bmatrix}$$

$$\boldsymbol{F}_\xi = \boldsymbol{R}\hat{F} \in \mathbb{R}^3 \qquad \hat{F} = \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^4 f_i \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} C\psi C\theta & C\psi S\theta S\phi - S\psi C\phi & C\psi S\theta C\phi + S\psi S\phi \\ S\psi C\theta & S\psi S\theta S\phi + C\psi C\phi & S\psi S\theta C\phi - C\psi S\phi \\ -S\theta & C\theta S\phi & C\theta C\phi \end{bmatrix}$$

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} \triangleq \begin{bmatrix} (f_3 - f_1)l \\ (f_2 - f_4)l \\ \sum_{i=1}^4 \tau_{M_i} \end{bmatrix}$$



Review

Translational lagrangian

$$L_{trans} = \frac{m}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}} - mgz$$

Translational equation

$$\frac{d}{dt}\left[\frac{\partial L_{trans}}{\partial \dot{\boldsymbol{\xi}}}\right] - \frac{\partial L_{trans}}{\partial \boldsymbol{\xi}} = \boldsymbol{F}_\xi$$

⇩

$$m\ddot{\boldsymbol{\xi}} + mg\boldsymbol{E}_z = \boldsymbol{F}_\xi$$

$$m\ddot{\xi} + mg\boldsymbol{E}_z = R\begin{bmatrix} 0 \\ 0 \\ \sum f \end{bmatrix}$$

Rotational lagrangian

$$L_{rot} = \frac{1}{2}\dot{\boldsymbol{\eta}}^T \mathbb{J} \dot{\boldsymbol{\eta}}$$

Rotational equation

$$\frac{d}{dt}\left[\frac{\partial L_{rot}}{\partial \dot{\boldsymbol{\eta}}}\right] - \frac{\partial L_{rot}}{\partial \boldsymbol{\eta}} = \boldsymbol{\tau}$$

⇩ $\frac{\partial L_{rot}}{\partial \dot{\boldsymbol{\eta}}} = \frac{1}{2}\left(\mathbb{J} + \mathbb{J}^T\right)\dot{\boldsymbol{\eta}} = \mathbb{J}\dot{\boldsymbol{\eta}}$

$$\mathbb{J}\ddot{\boldsymbol{\eta}} + \left(\dot{\mathbb{J}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^T \mathbb{J}\right)\right)\dot{\boldsymbol{\eta}} = \boldsymbol{\tau}$$

⇩ $C(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \dot{\mathbb{J}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^T \mathbb{J}\right)$

$$\mathbb{J}\ddot{\boldsymbol{\eta}} + C(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}} = \boldsymbol{\tau}$$

Review

$$\mathbb{J} = \begin{bmatrix} I_{xx} & 0 & -I_{xx}S\theta \\ 0 & I_{yy}C^2\phi + I_{zz}S^2\phi & (I_{yy}-I_{zz})C\phi S\phi C\theta \\ -I_{xx}S\theta & (I_{yy}-I_{zz})C\phi S\phi C\theta & I_{xx}S^2\theta + I_{yy}S^2\phi C^2\theta + I_{zz}C^2\phi C^2\theta \end{bmatrix}$$

$$C(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{31} & c_{33} \end{bmatrix}$$

$$c_{11} = 0$$
$$c_{12} = (I_{yy}-I_{zz})(\dot{\theta}C\phi S\phi + \dot{\psi}S^2\phi C\theta) + (I_{zz}-I_{yy})\dot{\psi}C^2\phi C\theta - I_{xx}\dot{\psi}C\theta$$
$$c_{13} = (I_{zz}-I_{yy})\dot{\psi}C\phi S\phi C^2\theta$$
$$c_{21} = (I_{zz}-I_{yy})(\dot{\theta}C\phi S\phi + \dot{\psi}S^2\phi C\theta) + (I_{yy}-I_{zz})\dot{\psi}C^2\phi C\theta + I_{xx}\dot{\psi}C\theta$$
$$c_{22} = (I_{zz}-I_{yy})\dot{\phi}C\phi S\phi$$
$$c_{23} = -I_{xx}\dot{\psi}S\theta C\theta + I_{yy}\dot{\psi}S^2\phi C\theta S\theta + I_{zz}\dot{\psi}C^2\phi S\theta C\theta$$
$$c_{31} = (I_{yy}-I_{zz})\dot{\psi}C\phi S\phi C^2\theta - I_{xx}\dot{\theta}C\theta$$
$$c_{32} = (I_{zz}-I_{yy})(\dot{\theta}C\phi S\phi S\theta + \dot{\phi}S^2\phi C\theta) + (I_{yy}-I_{zz})\dot{\phi}C^2\phi C\theta$$
$$\quad + I_{xx}\dot{\psi}C\theta S\theta - I_{yy}\dot{\psi}S^2\phi C\theta S\theta - I_{zz}\dot{\psi}C^2\phi C\theta S\theta$$
$$c_{33} = (I_{yy}-I_{zz})\dot{\phi}C\phi S\phi C^2\theta - I_{yy}\dot{\theta}S^2\phi C\theta S\theta - I_{zz}\dot{\theta}C^2\phi C\theta S\theta + I_{xx}\dot{\theta}C\theta S\theta$$

Passivity

$$N = \dot{\mathbb{J}} - 2\boldsymbol{C} = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$:skew symmetric

$$n_{11} = 0$$
$$n_{12} = 2(I_{zz}-I_{yy})(\dot{\theta}C\phi S\phi + \dot{\psi}S^2\phi C\theta) + 2(I_{yy}-I_{zz})\dot{\psi}C^2\phi C\theta + 2I_{xx}\dot{\psi}C\theta$$
$$n_{13} = -I_{xx}\dot{\theta}C\theta + 2(I_{yy}-I_{zz})\dot{\psi}C\phi S\phi C^2\theta$$
$$n_{21} = 2(I_{yy}-I_{zz})(\dot{\theta}C\phi S\phi + \dot{\psi}S^2\phi C\theta) + 2(I_{zz}-I_{yy})\dot{\psi}C^2\phi C\theta - 2I_{xx}\dot{\psi}C\theta = -n_{12}$$
$$n_{22} = 0$$
$$n_{23} = (I_{zz}-I_{yy})(\dot{\theta}C\phi S\phi S\theta + \dot{\phi}S^2\phi C\theta) + (I_{yy}-I_{zz})\dot{\phi}C^2\phi C\theta$$
$$\quad + 2I_{xx}\dot{\psi}S\theta C\theta - 2I_{yy}\dot{\psi}S^2\phi S\theta C\theta - 2I_{zz}\dot{\psi}C^2\phi S\theta C\theta$$
$$n_{31} = I_{xx}\dot{\theta}C\theta + 2(I_{zz}-I_{yy})\dot{\psi}C\phi S\phi C^2\theta = -n_{13}$$
$$n_{32} = (I_{yy}-I_{zz})(\dot{\theta}C\phi S\phi S\theta + \dot{\phi}S^2\phi C\theta) + (I_{zz}-I_{yy})\dot{\phi}C^2\phi C\theta$$
$$\quad - 2I_{xx}\dot{\psi}S\theta C\theta + 2I_{yy}\dot{\psi}S^2\phi S\theta C\theta + 2I_{zz}\dot{\psi}C^2\phi S\theta C\theta = -n_{23}$$
$$n_{33} = 0$$

Passivity

Passivity of transition

$$V_{trans} = \frac{1}{2}m\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}} + mgz \qquad z \geq 0$$

$$\dot{V}_{trans} = \frac{1}{2}m\ddot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}} + \frac{1}{2}m\dot{\boldsymbol{\xi}}^T\ddot{\boldsymbol{\xi}} + mg\dot{z}$$
$$= \dot{\boldsymbol{\xi}}^T m\ddot{\boldsymbol{\xi}} + mg\dot{z}$$
$$= \dot{\boldsymbol{\xi}}^T(\boldsymbol{F}_\xi - mg\boldsymbol{E}_z) + mg\dot{z}$$
$$= \dot{\boldsymbol{\xi}}^T\boldsymbol{F}_\xi = \boldsymbol{F}_\xi^T\dot{\boldsymbol{\xi}}$$

Passivity of rotation

$$V_{rot} = \frac{1}{2}\dot{\boldsymbol{\eta}}^T\mathbb{J}\dot{\boldsymbol{\eta}}$$

$$\dot{V}_{rot} = \frac{1}{2}\ddot{\boldsymbol{\eta}}^T\mathbb{J}\dot{\boldsymbol{\eta}} + \frac{1}{2}\dot{\boldsymbol{\eta}}^T\dot{\mathbb{J}}\dot{\boldsymbol{\eta}} + \frac{1}{2}\dot{\boldsymbol{\eta}}^T\mathbb{J}\ddot{\boldsymbol{\eta}}$$
$$= \dot{\boldsymbol{\eta}}^T\mathbb{J}\ddot{\boldsymbol{\eta}} + \frac{1}{2}\dot{\boldsymbol{\eta}}^T\dot{\mathbb{J}}\dot{\boldsymbol{\eta}}$$
$$= \dot{\boldsymbol{\eta}}^T(\boldsymbol{\tau} - C\dot{\boldsymbol{\eta}}) + \frac{1}{2}\dot{\boldsymbol{\eta}}^T\dot{\mathbb{J}}\dot{\boldsymbol{\eta}}$$
$$= \dot{\boldsymbol{\eta}}^T\boldsymbol{\tau} + \frac{1}{2}\dot{\boldsymbol{\eta}}^T(\dot{\mathbb{J}} - 2C)\dot{\boldsymbol{\eta}}$$
$$= \dot{\boldsymbol{\eta}}^T\boldsymbol{\tau} = \boldsymbol{\tau}^T\dot{\boldsymbol{\eta}}$$

Passivity

$$\int_0^T \boldsymbol{F}_\xi^T\dot{\boldsymbol{\xi}}dt = \int_0^T \dot{V}_{trans}dt$$
$$= V_{trans}(T) - V(0)_{trans}$$
$$\geq -V_{trans}(0) = -\beta_0$$

$$\int_0^T \boldsymbol{\tau}^T\dot{\boldsymbol{\eta}}dt = \int_0^T \dot{V}_{rot}dt$$
$$= V_{rot}(T) - V(0)_{rot}$$
$$\geq -V_{rot}(0) = -\beta_0$$

$\boldsymbol{F}_\xi$ ⇨ $\dot{\boldsymbol{\xi}}$

$\boldsymbol{\tau}$ ⇨ $\dot{\boldsymbol{\eta}}$