Progress Report : Stability Region Analysis for RD Model

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About RMD Model

Replicator Mutator Dynamics (RMD)[1]

$\boxed{\dot{\mathbf{x}} = Q^T F\mathbf{x} - \phi\mathbf{x}}$

x_i : proportion of population taking action i

$Q = I - \mu L$

$A = [a_{ij}]$: depend on the graph structure

$L = I - D^{-1}A$: fixed

$F = \mathrm{diag}(\mathbf{f})$ x_i : fraction

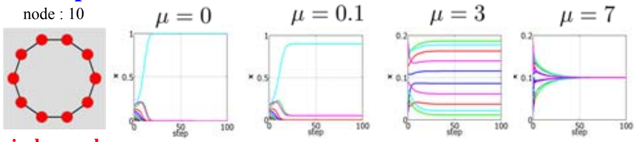
$\mathbf{f} = A\mathbf{x}$: Payoff

$\mu > 0$: mutation parameter (constant)

$\phi = \mathbf{f}^T\mathbf{x}$: average payoff

Example node : 10

① $\mu = 0$ ② $\mu = 0.1$ ③ $\mu = 3$ ④ $\mu = 7$



circle graph

About RMD Model

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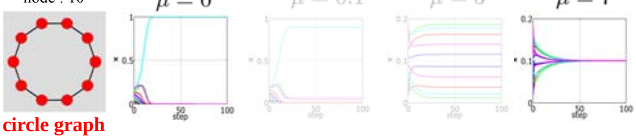


circle graph

Definition (*Consensus*)

A *Consensus* on RMD is defined as

$\bar{x}_i = 1$ and $\bar{x}_j = 0,\ \forall j \neq i$

in stable state $\bar{x}$.

Remark

All agents take one action.

Definition (*Complete Collapse*)

A *Complete Collapse* on RMD is defined as

$\bar{x}_i = 1/n,\ \forall i$

in stable state $\bar{x}$.

Remark

Ratios of agent taking each action are all same.

Stability Analysis for RMD Model

Proposition[1]

Let $\bar{x}$ be an equilibrium of the RMD. Then, the following statement hold.
A *consensus* results from $\mu = 0$.

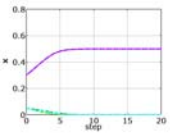
Proofs are omitting (refer to local report)

Problems

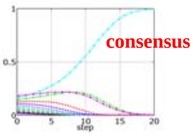
The mutation rate μ is only a necessary condition.

$\mu = 0$ Consensus

consensus $\rightarrow$ $\mu = 0$



$\mu = 0$



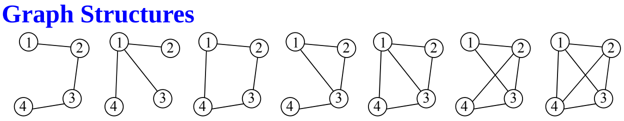
Simulation Example (node : 4)

Fixed Setting

Replicator Mutator Dynamics (RMD) $\dot{\mathbf{x}} = Q^T F\mathbf{x} - \phi\mathbf{x}$ $\xrightarrow{\mu = 0}$ Replicator Dynamics (RD) $\dot{\mathbf{x}} = F\mathbf{x} - \phi\mathbf{x}$

Node : 4

Graph Structures



Initial Values

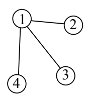
$\mathrm{x}_0 = [0.25\ 0.25\ 0.25\ 0.25]$
$\mathrm{x}_0 = [1\ 0\ 0\ 0]$
$\mathrm{x}_0 = [0.1\ 0.2\ 0.3\ 0.4]$
$\mathrm{x}_0 = [0.5\ 0.5\ 0\ 0]$
⋮

Graph Structures ✕ Initial Values

Correlation between initial value and convergence

Example

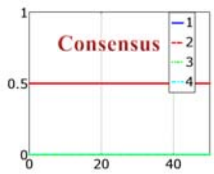
Graph Structure



$\mu = 0$

Initial Values

$\mathrm{x}_0 = [0.01\ 0.49\ 0.5\ 0]$
$\mathrm{x}_0 = [0.25\ 0.25\ 0.25\ 0.25]$
$\mathrm{x}_0 = [0.1\ 0.2\ 0.3\ 0.4]$
$\mathrm{x}_0 = [0.4\ 0.3\ 0.2\ 0.1]$
$\mathrm{x}_0 = [0.7\ 0.1\ 0.1\ 0.1]$
$\mathrm{x}_0 = [0.2\ 0.2\ 0\ 0.6]$
$\mathrm{x}_0 = [0.5\ 0.5\ 0\ 0]$



$\Rightarrow$ trajectory depends on the initial value

Orientation

Under a certain graph structure,
there must be the set of initial values achieving *consensus*

Region of Attraction (ROA) $\bar{x}_i = 1$ and $\bar{x}_j = 0,\ \forall j \neq 1$

All trajectory starting in this region will be attracted to the fixed point at the origin.

Example : Van Der Pole

Dynamics

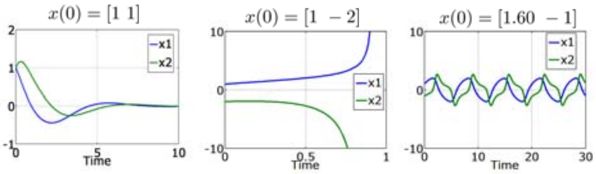
$\dot{x}_1 = -x_2$

$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$

unique equilibrium

$x = [0\ 0]$

$x(0) = [1\ 1]$ $x(0) = [1\ \ -2]$ $x(0) = [1.60\ \ -1]$

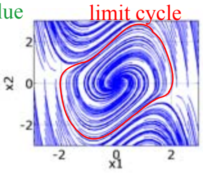


→ trajectory depends on the initial value

limit cycle

Phase Plane Plot

The unstable limit cycle forms the boundary between the convergent and divergent trajectories.



Region of Attraction

Definition

$\dot{x} = f(x(t))$: nonlinear dynamical system

$f(0) = 0$: the origin is equilibrium point

$\phi(\xi, t)$: solution to the system at time t with $\phi(\xi, 0)$

$\phi(\xi, 0) = \xi$: initial condition

Region of Attraction (ROA)

The ROA for the equilibrium point $x = 0$ of the system is

$$\{\xi \in \mathbf{R} : \lim_{t\to\infty} \phi(\xi, t) = 0\}.$$

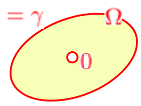
Estimation of the ROA Ω

$\Omega = \{x \in \mathbf{R}^n : V(x) \leq \gamma\}$

Conditions

$V(0) = 0, V(x) > 0$ for all nonzero $x \in \mathbf{R}^n$

$\Omega \backslash \{0\} \subset \{x \in \mathbf{R}^n : \nabla V(x) f(x) < 0\}$



Something called SOS is used as one solution for this problem.

Sum of Squares (SOS)

Definition

A polynomial p is an Sum of Squares (SOS) if there exist polynomials $g_1, \cdots, g_n$ such that $p = \sum_{i=1}^{n} g_i^2$.

Ex. $x_1^2 + 2x_1^4 + 2x_1^3 x_2 + -x1^2 x2^2 + 5x_2^4$ is an SOS since it can be expressed as

$x_1{}^2 + \frac{1}{2}(2x_1^2 - 3x_2^2 + x_1 x_2)^2 + \frac{1}{2}(x_2^2 + 3x_1 x_2)^2$ SOS polynomial is nonnegative everywhere.

(TOOLS :SOSTOOLS[2], YALMIP[3])

Estimation of the ROA with SOS

$\Omega = \{x \in \mathbf{R}^n : V(x) \leq \gamma\}$

$V(0) = 0, V(x) > 0,\ \forall \neq 0$

$\Omega \backslash \{0\} \subset \{x \in \mathbf{R}^n : \nabla V(x) f(x) < 0\}$ ⇨ $-(\varepsilon x^T x + \nabla V(x) f(x)) + s(V(x) - \gamma) \geq 0,\ \forall x$

$\begin{bmatrix} V, \varepsilon & \text{is given} \\ s & \text{is SOS} \end{bmatrix}$

$$\max \gamma \quad \begin{cases} s(x) \geq 0,\ \forall x \\ -(\varepsilon x^T x + \nabla V(x) f(x)) + s(V(x) - \gamma) \geq 0,\ \forall x \end{cases}$$

Application

Application to VDP

$\max \gamma$

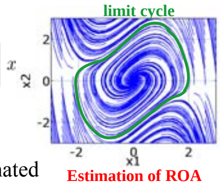
$s(x) \geq 0,\ \forall x$

$-(\varepsilon x^T x + \nabla V(x) f(x)) + s(V(x) - \gamma) \geq 0,\ \forall x$

$V = x^T \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} x$

$\varepsilon = 10^{-6}$

limit cycle



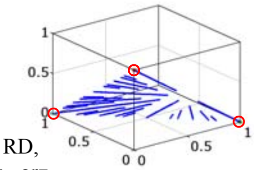
Estimation of ROA

⇨ some techniques to enlarge the estimated region of attraction are presented in [5]

Application to RD

① — ② — ③

$x_e = [1\ 0\ 0], [0\ 1\ 0], [0\ 0\ 1]$



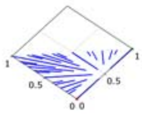
$\partial f / \partial x|_{x=x_e}$,which is the linearization of RD, have zero eigenvalues...orz

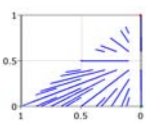
Future Works

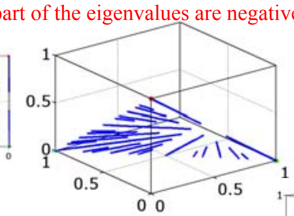
・increase the number of the node ⇨ cannot visualize it.

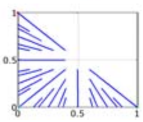
・consider about properties when $\mu \neq 0$.

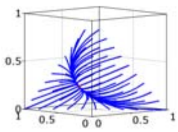
⇨ real part of the eigenvalues are negative











・explore the other approach to analyze the stability

Reference

[1] R. Olfati-Saber, "Evolutionary Dynamics of Behavior in Social Networks," in *Proc. of the 46th IEEE Conference on Decision and Control*, Dec. 2007, pp. 4051-4056.

[2] S. Prajna, A. Papachiristodoulou, P. Seiler and P. A. Parrilo, "SOSTOOLS : Sum of Squares Optimization Toolbox for MATLAB User's guide," 2002.

[3] 市原 裕之, "二乗和に基づく制御系解析・設計," システム /制御 /情報, Vol. 55, No.5 (2011).

[4] U.Topcu, A. Packard, P. Seiler and G. Balas, "Help on SOS," IEEE Control Systems Magazine, 30, 18–23 (2010).

[5] W. Tan and A. Packard, "Stability Region Analysis Using Composite Lyapunov Functions and Sum of Squares Programming," IEEE, Trans. Aitp,at. Contr, vol. 53, no.2, pp. 565—571, 2008.

Evolutionary Dynamics in Social Networks[8-12]

Behavior Network Topology

$G=(V,E,A)$ $V=\{b_1,\cdots b_n\}$ b_i : behavior (action)

connected $E=\{(b_i,b_j):a_{ij}>0\}$

undirected graph $A=[a_{ij}]$: adjacency matrix (assump. : $a_{ii}=1,\forall i$)

Replicator Mutator Dynamics (RMD)

$\boxed{\dot{\mathbf{x}}=Q^T F\mathbf{x}-\phi\mathbf{x}}$ $F=\mathrm{diag}(\mathbf{f})$ x_i : fraction

$Q=I-\mu L$: social choice model $\mathbf{f}=A\mathbf{x}$: Payoff

$\phi=\mathbf{f}^T\mathbf{x}$: average payoff

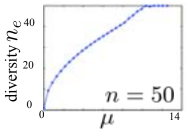
$L=I-D^{-1}A$: fixed $\mu>0$: mutation parameter (constant)

Diversity

how many species exist in steady state

$n_e=1/\sum x_i^2$

・roll of the structure of network and μ on diversity[10]



・small-world network improve the diversity on homogeneous networks[11]

Diversity*

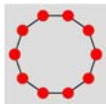
Phases for Evolution[8]

$n_e=1/\sum x_i^2$ $1\le n_e\le n$

Name	n_e	μ	$\mathbf{x}$
① Behavioral Flocking	1	0	$x_{i^*}=1, x_j=0, \forall j\ne i^*$
② Cohesion	$1<n_e\ll n$		
③ Collapse	$1\ll n_e<n$		
④ Complete Collapse	n	大	$x_i=1/n,\forall i$

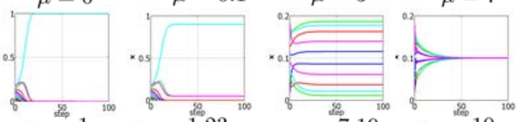
Example

node : 10



circle graph

① Behavioral Flocking $\mu=0$ ② Cohesion $\mu=0.1$ ③ Collapse $\mu=3$ ④ Complete Collapse $\mu=7$



$n_e=1$ $n_e=1.23$ $n_e=7.19$ $n_e=10$