Survey of Synchronization Part I: Kuramoto Oscillators

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FL11-5-2

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Outline of My Research in This Semester

- Survey of Synchronization
  - - Kuramoto oscillator : This Seminar
  - - Synchronization on SO(3) (SE(3)) : The Next Seminar
  - - Pursuit and Evasion : The 3rd Seminar
  - etc…

➡ Search new research fields or problem settings

- Visual Feedback Pose Synchronization
  - - Weaken the assumption where visibility structures are leader-follower type
  - - Search new procedures for proof

➡ Aim to submit a paper to the 51st CDC and ECC

- Study
  - - F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems*, Springer, 2004.
  - - M. Vidyasagar, *Nonlinear Systems Analysis, Second Edition*, SIAM, 2002
- Collaborative Work

Introduction

Synchronization

Multiple periodic processes with different natural frequencies come to acquire a common natural frequency as a result of their mutual or one-sided interaction

Collective Synchronization Phenomena

observed in Biological, Chemical and Social Systems

Networks of pacemaker cells in the heart
Circadian rhythms in the blain or living organisms
Synchronously flashing fireflies
Crickets chirping in unison etc…

Flashing Fireflies

Physics and Engineering

Arrays of lasers
Microwave oscillators
Computer clock synchronization
Superconducting Josephson junctions etc…

Josephson Junctions

> These numerous examples originally motivate researchers to study collective synchronization phenomena

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Outline

- Introduction
  - - Outline of My Research in This Semester
  - - Synchronization
- Survey of Kuramoto Oscillators
  - - History of Kuramoto Oscillator [1-4]
  - - Ali Jadbabaie et al. [5]
  - - Nikhil Chopra et al. [6-8]
  - - R. Sepulchre et al. [9,10]
  - - F. Dorfler et al. [11,12]
- ✕ Introduction to Visual Feedback Attitude Synchronization with Velocity Observers : Next Seminar
  - - Problem Settings
  - - Technical Difficulties
- Conclusion

History of Kuramoto Oscillators [1]

N. Wiener [2]

(motivated by the generation of alpha rhythms in the blain)

- first studied collective synchronization
- recognized its ubiquity in the natural world

> Unfortunately, his mathematical approach based on Fourier integrals has turned out to be a dead end

A. T. Winfree [3]

(motivated by circadian rhythms in living organisms)

- formulated the problem in terms of a huge population of interacting limit-cycle oscillators
- recognized that simplifications would occur if the coupling were weak and the oscillators nearly identical

> In simplifications, each oscillator is coupled to the collective rhythm generated by the whole population, analogous to a mean field approximation in physics

Winfee's Model

$$\dot{\theta}_i = \omega_i + \left(\sum_{j=1}^{N} X(\theta_j) \right) Z(\theta_i), \ i = 1, \cdots, N$$

θ_i : phase of oscillator i
ω_i : natural frequency of i
$X(\theta_j)$: phase-dependent influence on all the others
$Z(\theta_i)$: sensitivity function

History of Kuramoto Oscillators [1]

Y. Kuramoto [4]

- significantly extended Winfree's model
- recognized that the mean-field case should be the most tractable

> The long-term dynamics are given by the following phase equations corresponding to the simplest possible case of equally weighted, all-to-all, purely sinusoidal

Kuramoto Model

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \ i = 1, \cdots, N$$

$K \geq 0$: coupling gain

$1/N$ ensures that the model is well defined as $N \to \infty$

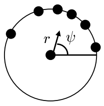
> To visualize the dynamics of the phases, it is convenient to imagine a swarm of points running around the unit circle in the complex plane

Order Parameter

(used by R. Sepulchre et al. [11,12])

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}$$

radius $r(t)$ measures the phase coherence
$\psi(t)$ is the average phase $(0 \leq r \leq 1)$



$r \approx 1$: the population acts like a giant oscillator $(\theta_i \approx \theta_j + 2\pi \cdot n)$
$r \approx 0$: no macroscopic rhythm is produced

History of Kuramoto Oscillators [1]

Kuramoto Model

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i),\ i = 1,\cdots,N$$

Order Parameter

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

$$re^{i\psi}\cdot e^{-i\theta_i} = re^{i(\psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}$$

Kuramoto Model

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i),\ i = 1,\cdots,N$$

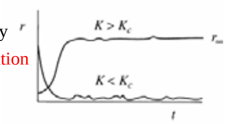
the phase θ_i is pulled toward the mean phase ψ

Simulations (how does $r(t)$ evolve?)

When $K > K_c$ for some $K_c > 0$, $r(t)$ grows exponentially

➡ mutually synchronized; generating a collective oscillation

When $K < K_c$, the oscillators act as if uncoupled;
the phases become distributed around the circle



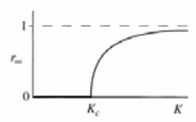
History of Kuramoto Oscillators [1]

By simulations, the author fond that

when $K > K_c$, the population splits into two groups;

i) The oscillators near the center of the frequency distribution lock together at the mean frequency Ω and co-rotate with the average phase $\psi(t)$
ii) Those in the tails run near their natural frequencies and drift relative to the cluster

Partially Synchronized



With the further increase in K, more and more oscillators are recruited into the synchronized cluster, and r_∞ grows

Main Results

If $N \to \infty$, then

$$K_c = \frac{2}{\pi g(0)},\ r_\infty \approx \sqrt{\frac{16}{\pi K_c^3}}\sqrt{\frac{K-K_c}{-g''(0)K_c}}$$

$g(\omega)$: probability density of ω_i

Ex.)

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2+\omega^2)} \Rightarrow r_\infty = \sqrt{1-\frac{K_c}{K}}$$

Sketch of Proof

$$\langle e^{i\theta}\rangle = \langle e^{i\theta}\rangle_{\mathrm{lock}} + \langle e^{i\theta}\rangle_{\mathrm{drift}} \Rightarrow$$

$\langle\cdot\rangle$: average

$\langle e^{i\theta}\rangle_{\mathrm{drift}} = 0$ for appropriate $g(\omega)$

$$\langle e^{i\theta}\rangle_{\mathrm{lock}} = Kr\int_{-\pi/2}^{\pi/2}\cos^2\theta g(Kr\sin\theta)d\theta = r$$

$r = 0$ at $K = K_c$ ➡ K_c ➡ r_∞

Questions

Kuramoto Model

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i),\ i = 1,\cdots,N$$

Main Results [4]

If $N \to \infty$, then

$$K_c = \frac{2}{\pi g(0)},\ r_\infty \approx \sqrt{\frac{16}{\pi K_c^3}}\sqrt{\frac{K-K_c}{-g''(0)K_c}}$$

Some Important Questions Associated with Kuramoto Oscillators

- How about finite N ?
- There is no analysis which shows that the oscillators in the Kuramoto model synchronize exponentially

Nobody has even touched the problems of global stability and convergence [1] (2000)

➡ Control and Graph Theoretic Methods [5-12]

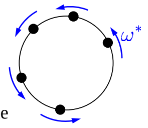
Hereafter,

Definition

The oscillators are said to synchronize if

$$\dot{\theta}_i - \dot{\theta}_j \to 0 \text{ as } t \to \infty\ {}^\forall i, j = 1,\cdots,N$$

i.e. the phase differences given by $\dot{\theta}_i - \dot{\theta}_j\ {}^\forall i, j = 1,\cdots,N$ become constant asymptotically



Preliminary: Incidence Matrix

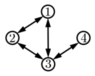
Graph Laplacian

$$L = D - A$$

D : degree matrix

A : adjacency matrix

Ex.)



$$A = \begin{bmatrix}0&1&1&0\\1&0&1&0\\1&1&0&1\\0&0&1&0\end{bmatrix}$$

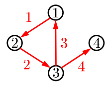
$$L = \begin{bmatrix}2&-1&-1&0\\-1&2&-1&0\\-1&-1&3&-1\\0&0&-1&1\end{bmatrix} \quad D = \begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&1\end{bmatrix}$$

Incidence Matrix $B \in \mathcal{R}^{N\times e}$

for Oriented Bidirectional Graph
with N Vertices and e Edges

$$B_{ij} = \begin{cases}1 & \text{if the edge is incoming to vertex } i\\ -1 & \text{if the edge is outcoming from vertex } i\\ 0 & \text{otherwise}\end{cases}$$

Ex.)



Oriented Bidirectional Graph

$$\Rightarrow B = N\left\{\overbrace{\begin{bmatrix}-1&0&1&0\\1&-1&0&0\\0&1&-1&-1\\0&0&0&1\end{bmatrix}}^{e}\right. \Rightarrow \boxed{BB^T = L}$$

$$\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i) \Rightarrow i\text{-th } \frac{K}{N}B\sin(B^T\theta)$$

$$(\theta = [\theta_1\ \cdots\ \theta_N]^T)$$

Kuramoto Model

$$\dot{\theta} = \omega + \frac{K}{N}B\sin(B^T\theta)$$

$$(\omega = [\omega_1\ \cdots\ \omega_N]^T)$$

A. Jadbabaie et al. [5]

Identical Coupled Oscillators

$\omega_i = \omega_s\ {}^\forall i = 1,\cdots,N$

Kuramoto Model

$$\dot{\theta} = \omega + \frac{K}{N}B\sin(B^T\theta)$$

Unperturbed Kuramoto Model

$$\dot{\theta} = \frac{K}{N}B\sin(B^T\theta) \qquad (\theta \to \theta - \omega_s t)$$

Theorem 1' [5]

Consider the unperturbed Kuramoto model. Then, for any given $\theta_0 \in \mathcal{R}$ and any value of $K > 0$, the vector $\theta_0\mathbf{1}$ is an asymptotically stable equilibrium solution, i.e., the synchronized state is globally asymptotically stable over $(-\frac{\pi}{2},\frac{\pi}{2})$.

Sketch of Proof

Lyapunov Function: $V = \frac{1}{2}\theta^T\theta$

$$\dot{V} = -\frac{K}{N}\theta^T\underline{L_w(\theta)\theta} \le 0$$

$L_w = B^T W(\theta) B$: weighted graph laplacian

$W(\theta) = \mathrm{diag}(\mathrm{sinc}(B^T\theta)_i)$

Lasalle's Invariance Principle ➡ $\theta \to \theta_0\mathbf{1}$ ➡ $||\dot{\theta}|| \to 0\ (\dot{\theta} \to \omega_s\mathbf{1})$

Theorem 1: Arbitrary connected graph

$(-\pi,\pi)$ ➡ The phase differences converge to an even multiple of 2π
i.e. the oscillators synchronize $(\dot{\theta}_i - \dot{\theta}_j = 0\ {}^\forall i, j = 1,\cdots,N)$

A. Jadbabaie et al. [5]

Non-identical Coupled Oscillators

Necessary Condition

Kuramoto Model

$$\dot{\theta} = \omega + \frac{K}{N}B\sin(B^T\theta)$$

$$(\omega = [\omega_1\ \cdots\ \omega_N]^T)$$

Order Parameter: $r^2 = \frac{1}{N^2}\left(N^2 - 2e + 2\mathbf{1}_e^T\cos(B^T\theta)\right)$

$$\frac{dr^2}{dt} = \frac{1}{N^2}\left(\frac{K}{N}(\sin(B^T\theta))^T B^T\underline{B\sin(B^T\theta)} - \omega^T\underline{B\sin(B^T\theta)}\right)$$

The critical value of the coupling is determined by the value of K for which the fixed point disappears.

$$\frac{dr^2}{dt} = 0 \Rightarrow \frac{N}{K}\omega = B\sin(B^T\theta^*)$$: fixed point equation

∞-norm: $\frac{N}{K}||\omega||_\infty = ||B\sin(B^T\theta^*)||_\infty \le ||B||_\infty = N-1$ ➡ $\boxed{K_L \ge \frac{N\omega_{max}}{N-1}}$

More technical calculations (cannot understand) …

Main Results

Necessary Condition

$$K_L \ge \frac{N(\omega_{max}-\omega_{min})}{2(N-1)}$$

If $K < K_L$, then a totally synchronized state does not exist

N. Chopra et al. [6,7]

Necessary Condition

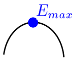
$$\dot\theta_i - \dot\theta_j = \omega_i - \omega_j + \frac{K}{N}\left(-2\sin(\theta_i-\theta_j) + \sum_{k=1,k\neq i,j}^{N} \sin(\theta_k-\theta_i) + \sin(\theta_j-\theta_k)\right) \quad (1)$$

Kuramoto Model

$$\dot\theta_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i),\ i=1,\cdots,N$$

Synchronization $\leftrightarrow \dot\theta_i - \dot\theta_j = 0$

$$\omega_j - \omega_i = \frac{K}{N}\underbrace{\left(2\sin(\theta_j-\theta_i) + \sum_{k=1,k\neq i,j}^{N}\sin(\theta_k-\theta_i)+\sin(\theta_j-\theta_k)\right)} := E_{ij}$$



By calculating $\frac{\partial E}{\partial\theta_i}, \frac{\partial E}{\partial\theta_j}, \frac{\partial E}{\partial\theta_k} = 0, \frac{\partial^2 E}{\partial\theta_m\theta_n} \le 0\ ^\forall m,n=1,\cdots,N$,

$$E_{ijmax} = 2\sin(\theta_j-\theta_i)_{opt} + 2(N-2)\sin\left(\frac{(\theta_j-\theta_i)_{opt}}{2}\right) \Rightarrow K_{cij} = \frac{(\omega_j-\omega_i)N}{E_{ijmax}}$$

The critical gain coupling gain required for onset of synchronization in (1) is given by

$$\boxed{K_c = \frac{(\omega_{max}-\omega_{min})N}{E_{max}}}$$

$$\boxed{[5]\quad K_L = \frac{(\omega_{max}-\omega_{min})N}{2(N-1)}}$$

Note: $K \ge K_c$ is a necessary condition below which synchronization cannot occur

$\boxed{[7]\ K_c = K_L\ N=2, N\to\infty,\ K_c > K_L \text{ for all other } N}$

N. Chopra et al. [6,7]

Sufficient Condition

$$\mathcal{D} = \{\theta_i,\theta_j \in \mathcal{R} \mid \underline{|\theta_i-\theta_j| \le \frac{\pi}{2}} - 2\epsilon\ ^\forall i,j=1,\cdots,N\} \qquad 0<\epsilon<\frac{\pi}{4}$$

Additional Assumption: $^\forall\theta_i \in \left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

Theorem 4.1 [6]

Consider the systems dynamics described by (1). Let all initial phase differences be contained in the compact set $\mathcal{D}$. Then, there exists a coupling gain $K_{inv} > 0$ such that $\underline{(\theta_i-\theta_j)\in\mathcal{D}\ ^\forall t>0}$.

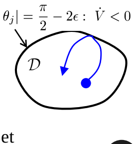
$$K_{inv} > \frac{N(\omega_{max}-\omega_{min})}{2\cos(2\epsilon)}$$

Positively Invariant Set

Sketch of Proof

Lyapunov Function: $V = \frac{1}{2K}(\theta_i-\theta_j)^2$

$$\dot V = \frac{1}{K}(\theta_i-\theta_j)(\dot\theta_i-\dot\theta_j) \le |\theta_i-\theta_j|\left|\frac{\omega_i-\omega_j}{K}\right| - \underbrace{(\theta_i-\theta_j)\sin(\theta_i-\theta_j)}_{\ge 0}\frac{2}{N}$$



If $K > \frac{N|\omega_i-\omega_j|}{2\cos(2\epsilon)}$, $\dot V$ is negative at $|\theta_i-\theta_j| = \frac{\pi}{2}-2\epsilon$

Thus, $\theta_i-\theta_j$ cannot leave $\mathcal{D}$ ⇒ K_{inv} : $\mathcal{D}$ is Positively Invariant Set

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N. Chopra et al. [6,7]

Main Results

$$\dot\theta_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i)$$

$$K_{inv} > \frac{N(\omega_{max}-\omega_{min})}{2\cos(2\epsilon)}$$

Theorem 4.2 [6]

If $K = K_{inv}$, then all the oscillators synchronize i.e. $\dot\theta_i - \dot\theta_j \to 0\ as\ t\to\infty\ ^\forall i,j=1,\cdots,N$.

Sketch of Proof Positive Function: $S = \frac{1}{2}\dot\theta^T\dot\theta$

$$\dot S = -\frac{K}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}\underbrace{\cos(\theta_i-\theta_j)}_{>0\ \text{(Theorem 4.1)}}(\dot\theta_i-\dot\theta_j)^2 \le 0$$

⇒ Lasalle's Invariance Principle

Corollary 4.3 [6]

If $K = K_{inv}$, then $\dot\theta_i \to \frac{\sum_{i=1}^{N}\omega_i}{N} = \Omega\ ^\forall i=1,\cdots,N\ as\ t\to\infty$.

Sketch of Proof

$$\sum_{i=1}^{N}\dot\theta_i = \sum_{i=1}^{N}\omega_i$$

Theorem 5.1 [6]

If $K = K_{inv}$, then the oscillators synchronize exponentially at a rate no worse that $\sqrt{K\sin(2\epsilon)}$.

Sketch of Proof (following [13]) $\dot\theta = \Omega\mathbf{1} + \delta$

δ : disagreement vector

$$\frac{d(\delta^T\delta)}{dt} \le -\frac{K}{N}\delta^T\sin(2\epsilon)\lambda_2(\underset{\text{Graph Laplacian}}{\underline{L}})\delta \le -K\sin(2\epsilon)\delta^T\delta$$

$\boxed{\text{All-to-all: } \lambda_2(L) = N}$

N. Chopra et al. [8]

Passivity-based Approach [8]

Kuramoto Model

$$\dot\theta_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i),\ i=1,\cdots,N$$

Case 1: $\omega_i = \omega_s\ ^\forall i=1,\cdots,N$

Transformation: $\Phi_i = \theta_i - \omega_s t\ ^\forall i$

⇒ $\begin{cases}\dot\Phi_i = u_i,\ u_i = \frac{K}{N}\sum_{j=1}^{N}\underline{\sin}(\Phi_j-\Phi_i) \\ y_i = \Phi_i\end{cases}$: passive with storage function $V_i = \frac{1}{2}\Phi_i^T\Phi_i$

satisfies odd function property. etc…

Assumption: $|\theta_i| \le \frac{\pi}{4}\ ^\forall i$

From Theorem 7 [8],

$$\boxed{\lim_{t\to\infty}|\Phi_j-\Phi_i| = 0\ ^\forall i,j \Rightarrow \lim_{t\to\infty}|\dot\theta_i-\omega_s| = 0\ ^\forall i}$$

Case 2: All oscillators have different natural frequencies

Derivative of the Kuramoto model: $\ddot\theta_i = \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\theta_i)(\dot\theta_j-\dot\theta_i),\ i=1,\cdots,N$

choose $\dot\theta_i = x_i,\ \cos(\theta_j-\theta_i) = g_{ji}(t)$

⇒ $\begin{cases}\dot x_i = u_i,\ u_i = \frac{K}{N}\sum_{j=1}^{N}\underline{g_{ji}(t)}\underbrace{(x_j-x_i)}_{y_j-y_i} \\ y_i = x_i\end{cases}$: passive with storage function $V_i = \frac{1}{2}x_i^Tx_i$

If $K = K_{inv}$, then $\mathcal{D}$ is positively invariant set. So $\underline{g_{ji}(t)} > 0$

regard as time variant positive gain

From Theorem 3 [8],

$$\boxed{\lim_{t\to\infty}|x_j-x_i| = 0\ ^\forall i,j\ \text{ i.e. } \lim_{t\to\infty}|\dot\theta_j-\dot\theta_i| = 0\ ^\forall i,j}$$

: Synchronization

R. Sepulchre et al. [9,10]

Autonomous Underwater Vehicles (AUVs)

$$\begin{cases}\dot r_k = e^{i\theta_k} \\ \dot\theta_k = u_k\end{cases},\ k=1,\cdots,N \quad (2)$$

Order Parameter

Phase Potential

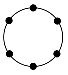
$$p_\theta = \frac{1}{N}\sum_{j=1}^{N}e^{i\theta_j}\ (=re^{i\psi}) \qquad U(\theta) = \frac{N}{2}|p_\theta|^2 \text{ or } \frac{N}{2}(1-|p_\theta|^2)$$

$$\Rightarrow u_k = -K\underbrace{\frac{\partial U}{\partial\theta_k}}_{\text{gradient of } U} = -K\underline{\langle p_\theta, ie^{i\theta_k}\rangle} = \underbrace{-\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_k)}_{\text{Kuramoto model without } \omega_k}$$

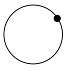
Theorem 1 [9]

The potential U reaches its unique minimum when $p_\theta = 0$ (balanced) and its unique maximum when all phases are identical (synchronization). All other critical points of U are isolated and saddle points of U.

almost global



balanced



synchronization

Sketch of Proof

$$\dot U = \frac{\partial U}{\partial\theta}\dot\theta = -K\sum_{k=1}^{N}\langle p_\theta, ie^{i\theta_k}\rangle^2 \Rightarrow \langle p_\theta, ie^{i\theta_k}\rangle = 0 :$$

$p_\theta = 0$: bal.

$p_\theta = e^{i\theta_k}$: sync.

$p_\theta = re^{i\psi}$: unstable

R. Sepulchre et al. [9,10]

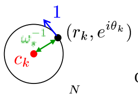
$$\dot\theta_k = u_k = \omega_s - K\frac{\partial U}{\partial\theta_k} = \omega_s - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_k) \Rightarrow \dot\theta = \omega_s\mathbf{1} - K\frac{\partial U}{\partial\theta}^T$$

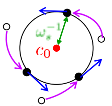
Since $\left\langle\frac{\partial U}{\partial\theta}^T, \mathbf{1}\right\rangle = 0$, the convergence analysis is unchanged

L_k : k-th row of L

Graph Laplacian: $\frac{\partial U}{\partial\theta_k} = \frac{K}{N}\langle ie^{i\theta_k}, L_ke^{i\theta}\rangle = \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_k)$

Spacing Control (Circular Formation)





Circular Formation

Center of the Circle: $c_k = r_k + i\omega_s^{-1}e^{i\theta_k}$

$s_k := -i\omega_s c_k = e^{i\theta_k} - i\omega_s r_k$

Lyapunov Function: $S(r,\theta) = \frac{1}{2}\|Ls\|^2$ ⇒ $\dot S = \langle Ls, L\dot s\rangle = \sum_{k=1}^{N}\langle L_ks, ie^{i\theta_k}\rangle\underline{(u_k-\omega_s)}$

$$\Rightarrow u_k = \omega_s - K\langle L_ks, ie^{i\theta_k}\rangle = \omega_s + K\frac{\partial U}{\partial\theta_k} + K\omega_s\langle\tilde r_k, \dot r_k\rangle \quad (3)$$

$(K>0)$ $\quad \tilde r_k = r_k - \frac{1}{N}\sum_{k=1}^{N}r_k$

Theorem 2 [9]

Consider the particle model (2) with the spacing control (3). All solutions converge to a relative equilibrium defined by a circular formation of radius ω_s.

Sketch of Proof

Lasalle's Invariance Principle

$$\dot S = -K\sum_{k=1}^{N}\langle L_ks, ie^{i\theta_k}\rangle^2 \le 0 \Rightarrow \langle L_ks, ie^{i\theta_k}\rangle = 0 \qquad \dot\theta_k = \omega_s,\ s = s_0\mathbf{1}$$

R. Sepulchre et al. [9,10]

Main Results

Theorem 3 [9]: Synchronized and Balanced Circular Formations

Consider the particle model (2). The following control law enforces convergence of all solutions to the set of relative equilibria defined by circularformations with a phase arrangement in the critical set of U.

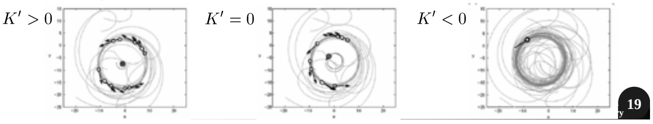
$$u_k = \omega_s + K\omega_s\langle\tilde{r}_k, \dot{r}_k\rangle + \frac{\partial(K-K')U}{\partial\theta_k}$$

Sketch of Proof

Lyapunov Function: $V(r,\theta) = KS(r,\theta) + U(\theta) = \frac{K}{2}\|Ls\|^2 + \frac{N}{2}|p_\theta|^2$

$$\dot{V} = -\sum_{k=1}^{N}\left(K\langle L_k s, ie^{i\theta_k}\rangle(u_k - \omega_s) + \frac{\partial U}{\partial\theta_k}u_k\right) = -\sum_{k=1}^{N}\left(K\langle L_k s, ie^{i\theta_k}\rangle + \frac{\partial U}{\partial\theta_k}\right)^2 \le 0$$

Lasalle ➡ $\dot{\theta}_k = \omega_s,\ s = s_0\mathbf{1},\ \underline{\langle p_\theta, ie^{i\theta_k}\rangle = 0}$ Almost: bal. or synch. (sign of K')



F. Dorfler et al. [11,12]

Smart Grid

The envisioned future power grid is expected to be even more complex and will rely increasingly on renewable energy sources, such as wind and solar power, which cause stochastic disturbances.

The future power grid is more prone to instabilities, which can ultimately lead to power blackouts.

Transient Stability

The ability of a power system to remain in synchronism when subjected to large transient disturbances such as faults on transmission elements or loss of load, generation, or system components.

➡ More general synchronization problem

Mathematical Model of a Power Network

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + P_{m,i} - P_{e,i},\ i = 1,\cdots,N$$

$$P_{e,i} = \sum_{j=1}^{N} E_iE_j(G_{ij}\cos(\theta_i - \theta_j) + B_{ij}\sin(\theta_i - \theta_j))$$

For each generator i ,

θ_i : rotor angle
$M_i > 0$: inertia
f_0 : 50 or 60 Hz
$D_i > 0$: damping constant
$P_{m,i} > 0$: mechanical power input
$P_{e,i}$: active power output
$E_i > 0$: internal voltage
$Y_{ij} = G_{ij} + \sqrt{-1}B_{ij}$: admittance
$(G_{ij} = G_{ji} \ge 0, B_{ij} = B_{ji} > 0)$

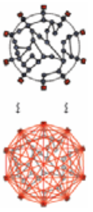
F. Dorfler et al. [11,12]

Mathematical Model of a Power Network

Natural Frequency: $\omega_i := P_{m,i} - E_i^2G_{ii}$

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum_{j=1}^{N} P_{ij}\sin(\theta_i - \theta_j + \phi_{ij})$$

$P_{ij} = E_iE_j|Y_{ij}|$
$P_{ii} := 0$
$\phi_{ij} = \tan^{-1}\left(\frac{G_{ij}}{B_{ij}}\right) \in [0, \pi/2)$



Inertia over damping ratio much smaller than the net frequency: $\frac{2M_i}{D_i} << 2\pi f_0$

Non-uniform Kuramoto Model

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j=1}^{N}\frac{P_{ij}}{D_i}\sin(\theta_i - \theta_j + \phi_{ij}) \quad (4)$$

Kuramoto Model

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

Main Results

Theorem 3.2 [11]

Consider the non-uniform Kuramoto model (4). Assume that

$$N\min_{i\neq j}\left\{\frac{P_{ij}}{D_i}\cos\phi_{ij}\right\} > \frac{1}{\cos\phi_{max}}\left(\max_{i\neq j}\left|\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} + 2\max_i\sum_{j=1}^{N}\frac{P_{ij}}{D_i}\sin\phi_{ij}\right|\right). \quad (5)$$

Then, for every $\theta(0) \in \Delta(\gamma_{max})$, the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{min}(0), \dot{\theta}_{max}(0)]$.

$\gamma_{max} \in \left(\frac{\pi}{2}, \pi\right]$

$\theta \in \Delta(\gamma)$ means $\max_{i,j}|\theta_i - \theta_j| < \gamma$

Sketch of Proof based on contraction theory (no details)

F. Dorfler et al. [11,12]

From Non-uniform to Classic

Theorem 3.2 [11]

Non-uniform Kuramoto Model

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j=1}^{N}\frac{P_{ij}}{D_i}\sin(\theta_i - \theta_j + \phi_{ij})$$

Remark 5.4 [11]

For the classic Kuramoto oscillators, the sufficient condition (5) of Theorem 3.2 specializes to

$$K > K_{critial} := \omega_{max} - \omega_{min}. \quad (6)$$

In other words, if $K > K_{critical}$, then the oscillators synchronize $(|\dot{\theta}_i - \dot{\theta}_j| = 0\ {}^\forall i, j)$.

Kuramoto Model

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

: Sufficient Condition

The authors claims that the condition (6) is also necessary condition since the bound (6) is close to the necessary conditions derived in [5-7].

Probably, the rigorous proof of Remark 5.4 is shown in [12], which is going to be presented in the 2011 American Control Conference.

Sufficient Condition in [6,7]

$$K_{inv} > \frac{N(\omega_{max} - \omega_{min})}{2\cos(2\epsilon)}$$

[5] $K_L = \frac{(\omega_{max} - \omega_{min})N}{2(N-1)}$

[6,7] $K_c = \frac{(\omega_{max} - \omega_{min})N}{E_{max}}$

Summary

Summary

Kuramoto Model: $\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i),\ i = 1,\cdots,N$

[2,3]: studied collective synchronization phenomena motivated by numerous examples in biological system, etc…

[4]: extended synchronization models in [3] and proposed the Kuramoto model, but said that the proof wasn't easy

} elegantly summarized in [1]

infinite N

⬇ Graph Theory

[5-9]: analyzed the necessary or sufficient conditions for synchronization finite N

[10-12]: considered applications by using the Kuramoto model

Graph theory also stimulates researchers to investigate the problem of coordinated motion of multiple autonomous agents (Cooperative Control)

Refer to [13] and references therein

Methods of Proofs

Most of works use nonnegaive phase potential whose derivative is nonpositive and investigate the equilibrium points. In analysis, the graph Laplacian plays a significant role.

Conclusions and Future Works

Attitude Synchronization

2D

$$\begin{cases} v_{wi}^b = v \\ \omega_{wi}^b = \underline{\omega_d + K_i\sum_{j\in\mathcal{N}_i} w_{ij}\sin(\theta_j - \theta_i)} \end{cases}$$

3D

$$\begin{cases} v_{wi}^b = v \\ \omega_{wi}^b = e^{-\hat{\xi}\theta_{wi}}\omega_d + K_i\sum_{j\in\mathcal{N}_i} w_{ij}\mathrm{sk}(e^{-\hat{\xi}\theta_{wi}}e^{\hat{\xi}\theta_{wj}})^\vee \end{cases}$$

like a Kuramoto model (Identical Coupled Oscillators)

Communication Graph directed, weighted, strongly connected, (switching)

Lyapunov Function Candidate

2D $U = \sum_{i=1}^{n}\frac{\gamma_i}{K_i}(1 - \cos\theta_i)$ 3D $U = \sum_{i=1}^{n}\frac{\gamma_i}{2K_i}\mathrm{tr}(I - e^{\hat{\xi}\theta_{wi}})$

Kuramoto Model in 3D

$$\begin{cases} v_{wi}^b = v \\ \omega_{wi}^b = e^{-\hat{\xi}\theta_{wi}}\omega_{di} + K\sum_{i=1}^{n}\mathrm{sk}(e^{-\hat{\xi}\theta_{wi}}e^{\hat{\xi}\theta_{wj}})^\vee \end{cases}$$

➡ $\|\omega_{wi} - \omega_{wj}\| = 0\ {}^\forall i, j$?

Assumption: $\bar{e}^{\hat{\xi}\theta_{wi}} > 0\ \left(|\bar{\theta}_{wi}| < \frac{\pi}{2}\right)$ ➡ The other equilibrium points are unstable ? [9,10]

The Next Survey: Synchronization on SO(3) (6/10)

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