Survey of Synchronization Part I: Kuramoto Oscillators

Tatsuya Ibuki
FL11-5-2
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Introduction

Synchronization
Multiple periodic processes with different natural frequencies come to acquire a common natural frequency as a result of their mutual or one-sided interaction.

Collective Synchronization Phenomena
observed in Biological, Chemical and Social Systems
- Circadian rhythms in the brain or living organisms
- Synchronously flashing fireflies
- Crickets chirping in unison etc...

Physics and Engineering
- Arrays of lasers
- Microwave oscillators
- Computer clock synchronization
- Superconducting Josephson junctions etc...

These numerous examples originally motivate researchers to study collective synchronization phenomena.

History of Kuramoto Oscillators [1]

N. Wiener [2] (motivated by the generation of alpha rhythms in the brain)
- first studied collective synchronization
- recognized its ubiquity in the natural world

Unfortunately, his mathematical approach based on Fourier integrals has turned out to be a dead end.

A. T. Winfree [3] (motivated by circadian rhythms in living organisms)
- formulated the problem in terms of a huge population of interacting limit-cycle oscillators
- recognized that simplifications would occur if the coupling were weak and the oscillators nearly identical

In simplifications, each oscillator is coupled to the collective rhythm generated by the whole population, analogous to a mean field approximation in physics.

Winfree’s Model
\[ \dot{\theta}_i = \omega_i + \frac{1}{N} \sum_{j=1}^{N} X(\theta_j) Z(\theta_i), \quad i = 1, \ldots, N \]
\( \omega_i \): phase of oscillator
\( \omega_i \): natural frequency of \( i \)
\( X(\theta_j) \): phase-dependent influence on all the others
\( Z(\theta_i) \): sensitivity function

Connection of History of Kuramoto Oscillators [1-4]

Y. Kuramoto [4]
- significantly extended Winfree’s model
- recognized that the mean-field case should be the most tractable

The long-term dynamics are given by the following phase equations corresponding to the simplest possible case of equally weighted, all-to-all, purely sinusoidal coupling:

Kuramoto Model
\[ \dot{\theta}_i = \omega_i + \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N \]
\( K > 0 \): coupling gain
**N** ensures that the model is well defined as \( N \to \infty \)

To visualize the dynamics of the phases, it is convenient to imagine a swarm of points running around the unit circle in the complex plane.

Order Parameter
\( r N = \frac{1}{N} \sum_{j=1}^{N} e^{i \theta_j} \) radius \( r(\theta) \) measures the phase coherence
\( \psi(\theta) \) is the average phase

**K** \( 0 \leq r \leq 1 \) \( r \approx 1 \): the population acts like a giant oscillator \( \theta_i \approx \theta_0 + 2\pi \cdot n \)

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Outline of My Research in This Semester

- **Survey of Synchronization**
  - Kuramoto oscillator
  - Synchronization on SO(3) (SE(3))
- **Pursuit and Evasion**
- **The Next Seminar**
- **The 3rd Seminar**

**Search new research fields or problem settings**

**Visual Feedback Pose Synchronization**
- Weaken the assumption where visibility structures are leader-follower type
- Search new procedures for proof

**Aim to submit a paper to the 51st CDC and ECC**

**Study**

**Collaborative Work**
Kuramoto Model
\[ \dot{\theta}_i = \omega + K \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N \]
Order Parameter
\[ \psi^{\text{osc}} = \frac{1}{N} \sum_{i=1}^{N} \psi_i = \frac{1}{N} \sum_{i=1}^{N} \sin(\theta_i - \theta) \]
Kuramoto Model
\[ \theta_i = \omega + K r \sin(\phi - \theta_i), \quad i = 1, \ldots, N \]

Simulations (how does \( r(t) \) evolve?)

When \( K > K_c \), for some \( K_c > 0 \), \( r(t) \) grows exponentially \( r \rightarrow \infty \), becoming synchronously \( \dot{\delta}(t) \rightarrow 0 \).

The phase \( \delta(t) \) is pulled toward the mean phase \( \bar{\theta} \).

With the further increase in \( K \), more and more oscillators are recruited into the synchronized cluster, and \( r \infty \) grows.

Some Important Questions Associated with Kuramoto Oscillators

• How about finite \( N \)?

• Unperturbed Kuramoto Model

\[ \dot{\theta}_i = \omega \quad \text{for} \quad \theta_i \in \mathbb{R} \]  

The oscillators near the center of the frequency distribution lock together at the mean frequency \( \bar{\theta} \) and drift relative to the cluster.

ii) Those in the tails run near their natural frequencies, and the phases become distributed around the circle.

Historically, the population splits into two groups; when \( K > K_c \), the population splits into two groups:

When \( K < K_c \), the oscillators act as if uncoupled; generating a collective oscillation.

Theorem 1:
When \( K < K_c \), the oscillators act as if uncoupled; when \( K > K_c \) for some \( K_c > 0 \), \( r(t) \) grows exponentially \( r \rightarrow \infty \), becoming synchronously \( \dot{\delta}(t) \rightarrow 0 \).

Theorem 1':
Kuramoto Model
\[ \dot{\theta}_i = \omega + K r \sin(\phi - \theta_i), \quad i = 1, \ldots, N \]

Ex.)

The phase differences converge to an even multiple of \( \pi \).

The oscillators synchronize exponentially if \( \theta_i \rightarrow \bar{\theta} \) i.e. the oscillators synchronize \( \dot{\theta}_i \rightarrow 0 \), \( i = 1, \ldots, N \)

Sketch of Proof

Lyapunov Function:
\[ V = \frac{1}{2} \theta^T \theta - V = - \frac{1}{2} \theta^T L_{\omega}(\theta^T) \theta \leq 0 \]

Lazare’s Invariance Principle:
\[ \theta \rightarrow \theta + \Lambda \]  

The phase differences converge to an even multiple of \( \pi \).

The oscillators synchronize \( \dot{\theta}_i \rightarrow 0 \), \( i = 1, \ldots, N \)
N. Chopra et al. [6,7]

**Main Results**

**Theorem 1** [9]

If $K = K_{osc}$, then all the oscillators synchronize i.e. $\theta_t - \theta_j \to 0$ as $t \to \infty$, $j = 1, \ldots, N$.

**Sufficient Condition**

$$D = \{\theta_0, \theta_j \in \mathbb{R} | \theta_t - \theta_j \leq \frac{\pi}{2}, j = 1, \ldots, N\}$$

$0 < \epsilon < \frac{\pi}{4}$

**Theorem 4.1** [6]

Consider the systems dynamics described by (1). Let all initial phase differences be contained in the compact set $D$. Then, there exists a coupling gain $K_{osc} > 0$ such that $[\theta_t - \theta_j] \in D$ for all $t > T$.

**Sketch of Proof**

Lyapunov Function: $V = \frac{1}{2K} (\theta_t - \theta_j)^2$

$$\dot{V} \leq \frac{1}{2K} (\theta_t - \theta_j) (\theta_t - \theta_j) - \frac{N}{2} \sin(\theta_t - \theta_j) \frac{2}{N} \leq 0$$

If $K > \frac{N}{2\cos(\pi/4)}$, $V$ is negative as $\theta_t - \theta_j = \frac{\pi}{2} - 2\epsilon > 0$.

Thus, $\theta_t - \theta_j$ cannot leave $D$.

**Corollary 4.3** [6]

$$\lim [\theta_t - \theta_j] = 0 \text{ for all } j \in [1, \ldots, N]$$

**Passivity-based Approach** [8]

**Case 1**

$$\dot{\theta}_i = \omega_0 + \frac{K}{N} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

**Transformation**

$$\Phi_i = \theta_i - \omega_0 \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

Passive storage function $V_1 = \frac{1}{2} \Phi_i^2$.

**Additional Assumption**

$$|\dot{\theta}_i| \leq \epsilon\pi > 0$$

**Theorem 7** [8]

$$\lim |\dot{\theta}_i| = 0 \text{ for all } i \in [1, \ldots, N]$$

**Passivity-based Approach** [8]

**Case 2**

$$\dot{\theta}_i = \omega_0 + \frac{K}{N} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

**Passive storage function**

$V_2 = \frac{1}{2} \Phi_i^2$.

**Additional Assumption**

$$|\dot{\theta}_i| \leq \epsilon\pi > 0$$

**Theorem 8** [8]

$$\lim |\dot{\theta}_i| = 0 \text{ for all } i \in [1, \ldots, N]$$

**Passivity-based Approach** [8]

**Case 3**

$$\dot{\theta}_i = \omega_0 + \frac{K}{N} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

**Passive storage function**

$V_3 = \frac{1}{2} \Phi_i^2$.

**Additional Assumption**

$$|\dot{\theta}_i| \leq \epsilon\pi > 0$$

**Theorem 9** [8]

$$\lim |\dot{\theta}_i| = 0 \text{ for all } i \in [1, \ldots, N]$$

**Passivity-based Approach** [8]

**Case 4**

$$\dot{\theta}_i = \omega_0 + \frac{K}{N} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

**Passive storage function**

$V_4 = \frac{1}{2} \Phi_i^2$.

**Additional Assumption**

$$|\dot{\theta}_i| \leq \epsilon\pi > 0$$

**Theorem 10** [8]

$$\lim |\dot{\theta}_i| = 0 \text{ for all } i \in [1, \ldots, N]$$

**Passivity-based Approach** [8]

**Case 5**

$$\dot{\theta}_i = \omega_0 + \frac{K}{N} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

**Passive storage function**

$V_5 = \frac{1}{2} \Phi_i^2$.

**Additional Assumption**

$$|\dot{\theta}_i| \leq \epsilon\pi > 0$$

**Theorem 11** [8]

$$\lim |\dot{\theta}_i| = 0 \text{ for all } i \in [1, \ldots, N]$$
Almost: bal. or synch. (sign of $K$).

Non-uniform Kuramoto Model

Kuramoto Model: $\dot{\theta}_i = \omega_i + K \sum_{j \neq i} \sin(\theta_i - \theta_j)$.

Main Results

Theorem 3 [9]: Synchronized and Balanced Circular Formations

Consider the particle model (2). The following control law enforces convergence of all solutions to the set of relative equilibria defined by circular formations with a phase arrangement in the critical set of $K$.

Sketch of Proof

Lyapunov Function: $V(\theta) = K S(\theta)$, where $S(\theta) = \sum_{i \neq j} \sin(\theta_i - \theta_j)$.

Sketch of Proof

Kuramoto Model: $\dot{\theta}_i = \omega_i + K \sum_{j \neq i} \sin(\theta_i - \theta_j)$.

Summary

Kuramoto Model: $\dot{\theta}_i = \omega_i + K \sum_{j \neq i} \sin(\theta_i - \theta_j)$, $i = 1, \ldots, N$.

Methods of Proofs

Most of works use nonnegative phase potential whose derivative is nonpositive and investigate the equilibrium points. In analysis, the graph Laplacian plays a significant role.

Conclusions and Future Works

Future Survey: Synchronization on $SO(3)$ (6/10)

Summary

Kuramoto Model: $\dot{\theta}_i = \omega_i + K \sum_{j \neq i} \sin(\theta_i - \theta_j)$.

Main Results

Theorem 3.1 [11,12]: Sufficient Condition

For the non-uniform Kuramoto model (4), assume that

\[ \dot{\theta}_i = \omega_i \pm K \sum_{j \neq i} \sin(\theta_i - \theta_j) \]

Then, for every $0 \leq \theta < 2\pi$, the frequencies $\dot{\theta}_i$ synchronize exponentially to some frequency $\omega$.

Sketch of Proof

Based on contraction theory (no details)

Sketch of Proof

Mathematical Model of a Power Network

Natural Frequency: $\omega_i = \omega_{s,i} - E_j G_{ij}$.

Inertia over damping ratio much smaller than the net frequency:

Theorem 3.2 [11]: Synchronized and Balanced Circular Formations

Kuramoto Model

From Non-uniform to Classic

Theorem 3.2 [11]: Sufficient Condition

The authors claims that the condition (6) is also necessary condition since the bound (6) is close to the necessary conditions derived in [5-7].

Conclusion: If $K > K_{\text{crit}}$, then the oscillators synchronize $|\theta_i - \theta_j| = 0$.

Summary

Kuramoto Model: $\dot{\theta}_i = \omega_i + K \sum_{j \neq i} \sin(\theta_i - \theta_j)$.

Main Results

Theorem 3.2 [11]: Sufficient Condition

For the classic Kuramoto oscillators, the sufficient condition (5) of Theorem 3.2 specializes to

\[ \dot{\theta}_i = \omega_i + K \sum_{j \neq i} \sin(\theta_i - \theta_j) \]

Kuramoto Model

Remark 5.4 [11]:

The future power grid is more prone to instabilities, which can ultimately lead to power blackouts.

Transient Stability

The ability of a power system to remain in synchronism when subjected to large transient disturbances such as faults on transmission elements or loss of load, generation, or system components.

For each generator $i$, $\theta_i$: rotor angle

Lyapunov Function Candidate

2D: $\dot{\theta}_i = \omega_i - E_i G_{ij} \sin(\theta_i - \theta_j)$.

Communication Graph

Directed, weighted, strongly connected, (switching)

Lyapunov Function Candidate

2D: $\dot{\theta}_i = \omega_i - E_i \sin(\theta_i - \theta_j)$.

Kuramoto Model in 3D

\[ \dot{\theta}_i = \omega_i - E_i \sin(\theta_i - \theta_j) + K \sum_{j \neq i} \sin(\theta_i - \theta_j) \]

Assumption: $\omega_i < \omega_j$.

The other equilibrium points are unstable.

Conclusions and Future Works

Attitude Synchronization

\[ \dot{\theta}_i = \omega_i + K \sum_{j \neq i} \sin(\theta_i - \theta_j) \]

3D: $\dot{\theta}_i = \omega_i - E_i \sin(\theta_i - \theta_j)$.

Lyapunov Function Candidate

2D: $U = \sum_{i=1}^{N} \frac{1}{K_i} (1 - \cos(\theta_i))$.

Kuramoto Model in 3D

\[ \dot{\theta}_i = \omega_i - E_i \sin(\theta_i - \theta_j) + K \sum_{j \neq i} \sin(\theta_i - \theta_j) \]

Assumption: $\omega_i < \omega_j$.

The other equilibrium points are unstable.
References