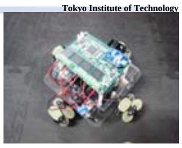
Survey of Camera Network Localization

Takayuki Nishi
FL11-4-2
11th, May, 2011

Overview of Localization



Mobile Robot Localization

Determining the pose of a robot relative to a given map of the environment

Knowledge of the location is necessary in robotic tasks

- Navigation
- Surveillance
- Mapping

Sensors for detect the location

- Computer-vision
- GPS
- Encoder
- Range finder etc.

Localization Problems

Consideration

- Pose can usually not sensed directly
- No noise-free sensor
- Dynamic environment

⇨ Integrate data to determine the pose

Taxonomy of localization problems

- Local localization (position tracking)
  Initial robot pose is known
- Global localization
  Initial robot pose is unknown Robot is initially placed
- Kidnapped robot problem
  During operation, the robot gets teleported

Multi-agent localization

Multi-agent localization (cooperation)

Each robot localize itself individually

⇩

Robots can detect each other (Relative location of robots)

Localization in sensor networks (Network localization)

GPS
- Not work well in buildings or obstruction
- High cost

Special nodes (beacon) Know position

Relative pose estimation for pair of robots only distance measurements [2]

Heuristic-based (lack of theoretical foundation)

[2] N. Trwmy, X. S. Zhou, K. X. Zhou and S. I. Roumeliotis, "3D Relative Pose Estimation from Distance-only Measurements," *Proc. of the 2007 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pp. 1071-1078, 2007.

Theory of Network Localization

Theory of network localization [3]

1. What are the condition for unique network localizability?
2. What is the computational complexity of network localization?
3. What is the complexity of network localization in typical network deployment scenarios?

Localization theory in angle-of-arrival sensing [4]

- Formulation of frame localizability
- Distributed algorithm for planar orientation localization
- Conditions for orientation localizability in noiseless 3-D networks

Francesco Bullo

[4] G. Piovan, I. Shames, B. Fidan, F. Bullo and B. D. O. Anderson, "On Frame and Orientation Localization for Relative Sensing Networks," Automatica, 2011. (submitted)

Localization in Camera Networks

Localization in camera networks [5]

Distributed estimation of the poses of the cameras

- Estimated neighboring poses by standard computer vision techniques
- Minimizing a cost function on SE(3) in a distributed fashion

R. Vidal

Related work

- Pose Averaging [6], [7]

[5] R. Tron and R. Vidal, "Distributed Image-based 3-D Localization of Camera Sensor Networks," *Proc. of the 48th IEEE Conference on Decision and Control*, pp. 901-908, 2009.

[6] R. Tron, R. Vidal and A. Terzis, "Distributed Pose Averaging in Camera Sensor Networks via Consensus on SE(3)," *Proc. of the International Conference on Distributed Smart Cameras*, 2008.

[7] T. Hatanaka, T. Nishi and M. Fujita, "Passivity-based Cooperative Estimation Algorithm for Networked Visual Motion Observers," *Proc. of the SICE Annual Conference 2011*, 2011. (submitted)

Comparison of Related Work

Visual sensor networks

- Static object and scene

R. Vidal

[6] ⇒ [5]

Pose averaging — Localization

[7] ⇒ My work??

Fujita Lab.

- Dynamic Estimation
- Simultaneous Estimation & Optimization

Outline

- Introduction
- Survey of [5]
- Application to our Research

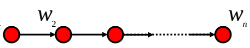
Preliminaries

Undirected connected graph $G=(V,E)$

Set of nodes: $V=\{1,\cdots,N\}$ Edge $(i,j)\in E$: j communicate with i

Pose of each node: $g_i=(R_i,T_i)\in SE(3)$ Relative Pose: $g_{ij}=g_i^{-1}\circ g_j$

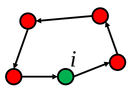
Path l : sequences of nodes



$l=\{w_1,\cdots,w_n\}, w_m\in V, (w_m - w_{m+1})\in E, m\in\{1,\cdots,n-1\}$

Relative pose along a path l: $g_l = g_{w_n w_{n-1}}\circ\cdots\circ g_{w_2 w_1}$

Cycle: Path from node i to itself without repeated nodes



Definition 1 (Localized network) [5]

A network is said to be localized if there is a set of relative transformations g_{ij} such that, when the reference frame of the first node is fixed to g_1, the other absolute poses g_i are uniquely determined.

Method to Get Noisy Estimates

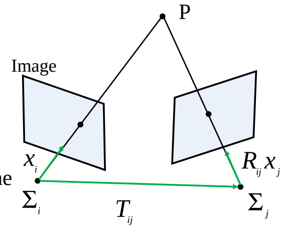
Epipolar Constraint

Extract feature points x_i, x_j

x_i, x_j : homogeneous coordinates

$x=[a \quad b \quad 1]^T$

Vector $x_i, T_{ij}, R_{ij}x_j$ lie in an epipolar plane



$x_i^T \hat{T}_{ij} R_{ij} x_j = 0$

Use more than 8 feature points ⇒ derive $t_{ij}=\frac{T_{ij}}{\|T_{ij}\|}, R_{ij}$

Measurements are corrupted by noise

Unknown scales λ_{ij} $\tilde{T}_{ij}=\lambda_{ij}\tilde{t}_{ij}$

Noisy relative pose $\tilde{g}_{ij}=(\tilde{R}_{ij},\tilde{T}_{ij})$

Problem Settings

Goal: Find the relative transformations g_{ij}

1. Close as the relative measurements
2. Satisfy the constraints given in Definition1

① Minimize the cost function

$$\varphi=\frac{1}{2}\sum_{j\in\mathcal{N}_i} d_g^2(g_{ij},\tilde{g}_{ij})$$

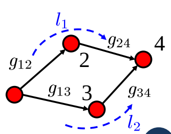
$$d_g^2(g_1,g_2)=d_{SO(3)}^2(R_1,R_2)+\|T_1-T_2\|^2$$

$$d_{SO(3)}^2(R_1,R_2)=-\frac{1}{2}\mathrm{trace}(\log(R_1^T R_2)^2)$$

Neighbor set: $\mathcal{N}_i=\{j\in V|(j,i)\in E\}$

② Constraints

For any path l from node 1 to node i, we have $g_i = g_l \circ g_1$, regardless of chosen path[1]



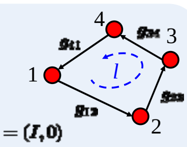
Consideration of the Consistent Constraints

Reparametrize relative pose g_{ij}

Using absolute pose g_i, g_j

Proposition 1 : The following are equivalent

1. The network is localized
2. For any cycle $l=\{w_1,\cdots,w_n,w_1\}, w_m\in V, n>1$ The transformation along the cycle is $g_l=(I,0)$
3. There exist a set of absolute poses g_i such that $g_{ij}=g_i^{-1}\circ g_j$



Compute the absolute poses g_i, relative pose g_{ij} will be uniquely determined

⇓

The consistent constraints will be automatically satisfied

Problem Settings

Rewrite using absolute poses

$$\varphi = \frac{1}{2}\sum_{j\in\mathcal{N}_i} d_g^2(g_{ij},\tilde{g}_{ij}) = \frac{1}{2}\sum_{j\in\mathcal{N}_i} d_g^2(g_i^{-1}g_j,\tilde{g}_{ij})$$
$$= \frac{1}{2}\sum_{j\in\mathcal{N}_i}(d^2_{SO(3)}(R_i^T R_j,\tilde{R}_{ij}) + \|R_i^T(T_j - T_i) - \lambda_{ij}\tilde{t}_{ij}\|^2)$$
$$= \underline{\varphi_R(\{R_i\})} + \underline{\varphi_T(\{R_i\},\{T_i\},\{\lambda_{ij}\})}$$

Only Rotation — All variables

Consider $\lambda_{ij}=0$

$\lambda_{ij}=0$ ⇨ $T_i=T_j$ $\varphi_T=0$ Global minimum but trivial

λ_{ij} must be positive ⇨ Proposed idea: Minimum scale $\lambda_{ij}\geq 1$

Summary (localization problem)

$\min_{\{R_i\},\{T_i\},\{\lambda_{ij}\}} \varphi(\{R_i\},\{T_i\},\{\lambda_{ij}\})$ subject to $\lambda_{ij}\geq 1$ $(i,j)\in E$

Procedure of Minimization

Procedure of minimization (gradient descent)

1. Find an initial set of rotations by optimizing only φ_R

$$\text{grad}_{R_k}\varphi_R = -R_k \sum_{i\in\mathcal{N}_k} \log(R_k^T R_i \tilde{R}_{ki}^T) + \log(R_k^T R_i \tilde{R}_{ik})$$

$$R_k(l+1) = \exp_{R_k(l)}(-\epsilon\,\text{grad}_{R_k(l)})$$

$R_k(l)$: Rotation at iteration l

ε : Step size

Initialization of the Rotations — φ_R has multiple local minima

Set new cost function

$$\varphi_R' = \frac{1}{2}\sum_{j\in\mathcal{N}_i}\|R_j - R_i\tilde{R}_{ij}\|_F^2$$

φ_R' doesn't have local minima

Find $R_i(0)$ by minimizing new cost function

$$\frac{\partial\varphi_R'}{\partial R_k} = \sum_{i\in\mathcal{N}_k}(R_k - R_i\tilde{R}_{ki}^T) + \sum_{k\in\mathcal{N}_i}(R_k - R_i\tilde{R}_{ik})$$

Procedure of Minimization

Procedure of minimization (gradient descent)

2. Find an initial set of translation and scales by optimizing only φ_T

$\min \varphi_T$ subject to $\lambda_{ij}\geq 1 (i,j)\in E$ — Rotation R_i : fixed

$$\frac{\partial\varphi_T}{\partial T_k} = 2\sum_{i\in\mathcal{N}_k}(T_k - T_i) + \lambda_{ki}R_k\tilde{t}_{ki} - \lambda_{ik}R_i\tilde{t}_{ik}$$
$$\frac{\partial\varphi_T}{\partial\lambda_{lk}} = \lambda_{lk} - (T_l - T_k)^T R_k\tilde{t}_{kl}$$

Update procedure

$$T_k(l+1) = T_k(l) - \epsilon\frac{\partial\varphi_T}{\partial T_k}$$
$$\lambda_{lk}(l+1) = \max\{1, \lambda_{lk}(l) - \epsilon\frac{\partial\varphi_T}{\partial\lambda_{lk}}\}$$

Global optima of φ_T ($\lambda_{ij}=0$) are not in the feasible set

One of the constraints need to be active $\min_{(i,j)\in E}\lambda_{ij}=1$

Procedure of Minimization

Procedure of minimization (gradient descent)

3. Optimize φ over all the variables

$$\text{grad}_{R_k}\varphi = \text{grad}_{R_k}\varphi_R + \sum_{i\in\mathcal{N}_k}\lambda_{ki}((T_i - T_k)\tilde{t}_{ki}^T - R_k\tilde{t}_{ki}(T_i - T_k)^T R_k)$$

$$\frac{\partial\varphi}{\partial T_k} = \frac{\partial\varphi_T}{\partial T_k} \qquad \frac{\partial\varphi}{\partial\lambda_{lk}} = \frac{\partial\varphi_T}{\partial\lambda_{lk}}$$

Reason of multi-step optimization

φ has multiple local minima ⇨ Good initialization is needed

Choice of step size

In gradient descent, it is important to select

Outline

- Introduction
- Survey of [5]
- Application to our Research

Preliminaries

Pose of camera i relative to world frame

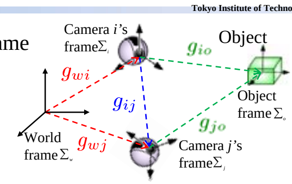
$g_{wi} = (p_{wi}, R_{wi})$

Pose of object relative to camera i

$g_{io} = (p_{io}, R_{io})$

Relative pose of the camera i and j

$g_{ij} = (p_{ij}, R_{ij})$



Estimated by Visual Motion Observer [8]

$\bar{g}_{io}$:Estimated pose of object by camera i

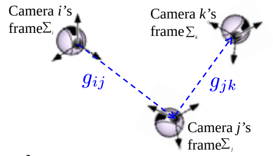
Definition of localization

Given a relative sensing network with reference node1. The reference frame transformation g_{1i} for all $i\in\{2,\cdots,n\}$ are uniquely determined.

Measurement Methods

Relative sensing by cameras

- One camera sense relative pose of another camera (visibility graph)



Attach feature points to camera

Directed graph $G=(V,E)$ Set of nodes: $V=\{1,\cdots,N\}$

Edge $(i,j)\in E$: i measure j's relative pose

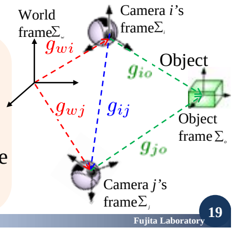
- Use object to estimate the pose

$\bar{g}_{ij} = \bar{g}_{io}\bar{g}_{jo}^{-1}$

$\bar{g}_{ij}$:Estimated relative pose

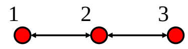
Edge $(i,j)\in E$: j measure i's relative pose

Consider undirected connected graph



Problem Settings

Consider 3 cameras (for simplicity)



New estimate variables: $\tilde{g}_{ij}$

Measurements (estimated by VMO): $\bar{g}_{ij}$

What we want to do is minimize the overall cost function

$\varphi(\tilde{g}_{12},\bar{g}_{12})+\varphi(\tilde{g}_{23},\bar{g}_{23})+\varphi(\tilde{g}_{31},\bar{g}_{31})$ in a distributed way

Close as the measurements

with a constraints $\tilde{g}_{12}\tilde{g}_{23}\tilde{g}_{31}=(I,0)$

Determine the pose

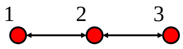
Cost function $d_g^2(g_1,g_2)=d_{SO(3)}^2(R_1,R_2)+\|T_1-T_2\|^2$

From [5] (Geodesic distance) $d_{SO(3)}^2(R_1,R_2)=-\frac{1}{2}\mathrm{trace}(\log(R_1^TR_2)^2)$

From [7] (Euclidean distance) $\varphi(g_1^{-1}g_2)=\|p_1-p_2\|^2+\frac{1}{2}\|R_1-R_2\|_F^2$

Problem Settings

Reparametrization



$\tilde{g}_{ij}=\tilde{g}_{wi}^{-1}\tilde{g}_{wj}$ $\tilde{g}_{wi},\tilde{g}_{wj}$:Absolute poses

Reference: node1 $\tilde{g}_{w1}=(I,0)$

Rewrite cost function

$\varphi(\tilde{g}_{1o},\bar{g}_{1o})+\varphi(\tilde{g}_{2o},\bar{g}_{2o})+\varphi(\tilde{g}_{3o},\bar{g}_{3o})+\varphi(\tilde{g}_{w1}\tilde{g}_{1o},\tilde{g}_{w2}\tilde{g}_{2o})+\varphi(\tilde{g}_{w2}\tilde{g}_{2o},\tilde{g}_{w3}\tilde{g}_{3o})$

Close as the measurements   Constraints (Determine the pose)

The pose of the object g_{wo} should be same, regardless of chosen path

We fix two variables $\tilde{g}_{w1}=(I,0),\tilde{g}_{w3}$

⇨ The solution will be trivial $\tilde{g}_{io}=\bar{g}_{io}$ $\tilde{g}_{wi}\bar{g}_{io}=\tilde{g}_{wj}\bar{g}_{jo}$

Decision variables $\tilde{g}_{1o},\tilde{g}_{2o},\tilde{g}_{3o},\tilde{g}_{w2}$

Gradient Method

First we consider only orientation

Minimize the cost function $\min_{\tilde{R}_{1o},\tilde{R}_{2o},\tilde{R}_{3o},\tilde{R}_{w2}}\Phi$

$$\Phi=\frac{1}{2}\|\tilde{R}_{1o}-\bar{R}_{1o}\|_F^2+\frac{1}{2}\|\tilde{R}_{2o}-\bar{R}_{2o}\|_F^2+\frac{1}{2}\|\tilde{R}_{3o}-\bar{R}_{3o}\|_F^2$$
$$+\frac{1}{2}\|\tilde{R}_{w1}\tilde{R}_{1o}-\tilde{R}_{w2}\tilde{R}_{2o}\|_F^2+\frac{1}{2}\|\tilde{R}_{w2}\tilde{R}_{2o}-\tilde{R}_{w3}\tilde{R}_{3o}\|_F^2$$

Gradient descent

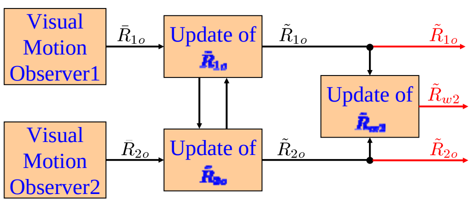
Calculate the gradient $\frac{\partial\Psi}{\partial\tilde{R}_{1o}}$ $\frac{\partial\Psi}{\partial\tilde{R}_{2o}}$ $\frac{\partial\Psi}{\partial\tilde{R}_{3o}}$ $\frac{\partial\Psi}{\partial\tilde{R}_{w2}}$

Update the estimates $\tilde{R}_{1o}(l+1)=\exp_{\tilde{R}_{1o}(l)}(-\epsilon\mathrm{grad}_{\tilde{R}_{1o}(l)}\Psi)$

$\tilde{R}_{1o}(l)$: Estimates at iteration l ε : Step size

Block Diagram of the Update Procedure

With 2 cameras



Future Works

- Test and simulation
- Settings of the cost function (Interpretation)
- Consideration of directed graph

References

[1] S. Thrun, W. Burgard and D. Fox, "Probabilistic Robotics," The MIT Press, 2006.

[2] N. Trwmy, X. S. Zhou, K. X. Zhou and S. I. Roumeliotis, "3D Relative Pose Estimation from Distance-only Measurements," *Proc. of the 2007 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pp. 1071-1078, 2007.

[3] J. Aspnes, T. Eren, D. K. Goldenberg, A. S. Morse, W. Whiteley, Y. R. Yang, B. D. O. Anderson and P. N. Belhumeur, "A Theory of Network Localization," *IEEE Transactions on Mobile Computing*, Vol. 5, No. 12, pp. 1663-1678, 2006.

[4] G. Piovan, I. Shames, B. Fidan, F. Bullo and B. D. O. Anderson, "On Frame and Orientation Localization for Relative Sensing Networks," Automatica, 2011. (submitted)

[5] R. Tron and R. Vidal, "Distributed Image-based 3-D Localization of Camera Sensor Networks," *Proc. of the 48th IEEE Conference on Decision and Control*, pp. 901-908, 2009.

References

[6] R. Tron, R. Vidal and A. Terzis, “Distributed Pose Averaging in Camera Sensor Networks via Consensus on SE(3),” *Proc. of the International Conference on Distributed Smart Cameras*, 2008.

[7] T. Hatanaka, T. Nishi and M. Fujita, “Passivity-based Cooperative Estimation Algorithm for Networked Visual Motion Observers,” *Proc. of the SICE Annual Conference 2011*, 2011. (submitted)

[8] M. Fujita, H. Kawai and M. W. Spong, “Passivity-based Dynamic Visual Feedback Control for Three Dimensional Target Tracking: Stability and L2-gain Performance Analysis,” *IEEE Transactions on Control Systems Technology*, Vol.15, No. 1, pp. 40-52, 2007.

[8] P. A. Absil, R. Mahony and R. Sepulchre, “Optimization Algorithms on Matrix Manifolds,” Princeton Press, 2008.

Appendix

Contributions of [4]

Contributions

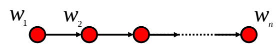
1. Formulation of frame localizability
2. Characterization of frame localizability for planar networks
3. Compute least-squares estimate of the orientations in a 2-D ring network
4. Distributed algorithm for planar orientation localization
5. Conditions for orientation localizability in noiseless 3-D networks

Elements of Graph Theory[4]

Directed or undirected graph $G=(V,E)$

Set of nodes: $V=\{1,\cdots,N\}$ Edge: $(i,j)\in E$

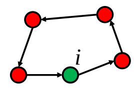
Path P: sequences of nodes



$P=\{w_1,\cdots,w_n\}, w_m\in V, (w_m - w_{m+1})\in E, m\in\{1,\cdots,n-1\}$

Cycle : $l\in\mathcal{L}(G_d)$ G_d : directed graph

Path from node i to itself without repeated nodes



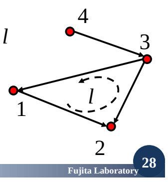
Cycle vector $\mathbf{1}_l\in\{-1,0,1\}^m\subset\mathbb{R}^m$

1 : orientation is consistent with the orientation of l

-1 : orientation is opposite with the orientation of l

0 : otherwise

Ex.) $\mathbf{1}_l=\begin{bmatrix}1 & -1 & 1 & 0\end{bmatrix}^T$

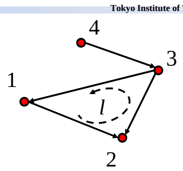


Elements of Graph Theory[4]

Set of cycle vectors $L=\{\mathbf{1}_l|\forall l\in\mathcal{L}(G_d)\}$

Set of fundamental cycle vectors $L_f\subseteq L$

Constitute a base for L



Fundamental cycles $\mathcal{L}_f(G)=\{l\in\mathcal{L}(G_d)|\mathbf{1}_l\in L_f\}$

Cycle matrix

$C=[\mathbf{1}_{l_1},\cdots,\mathbf{1}_{l_k}]^T$ k: Cardinality of L

Fundamental cycle matrix

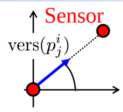
$C_f=[\mathbf{1}_{l_1},\cdots,\mathbf{1}_{l_r}]^T$ for all $\mathbf{1}_{l_i}\in L_f$ $r=\dim(L_f)$

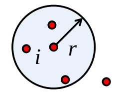
Full rank matrix and not unique

Localization Problem[4]

Angle-of-arrival sensing

Measurement of node i for node j





$\mathrm{vers}(p_j^i)\in\mathbb{R}^d$ $\mathrm{vers}(\mathbf{v})=\frac{\mathbf{v}}{\|\mathbf{v}\|}(\mathbf{v}\neq 0)$

Sensing range: r Sensing graph (Undirected graph): $G_s=(V_s,E_s)$

Remark 8 (Data referencing motivation)
Measurements are taken in their respective reference frames

Definition of localization

Problem 6 (Frame localizability)
Given a relative sensing network with reference node1.
The reference frame transformation $\{R_i^1, p_i^1\}$ for all $i\in\{2,\cdots,n\}$ are uniquely determined by the relative measurements

R_i^1 : i’s orientation relative to node1 p_i^1 : i’s position

Orientation Localizability[4]

Relationships $\mathrm{vers}(p_j^i) = -R_j^i \mathrm{vers}(p_i^j)$ (7)

Lemma 10 (Feasible orientations)

Compute $u_j^i = \mathrm{vers}(p_j^i)$ $\mathbf{H}_j^i = \exp(\alpha_j^i \hat{\mathbf{e}}_j^i)$ $\alpha_j^i \in [0, \pi]$

$$\mathbf{e}_j^i = \begin{cases} \mathrm{vers}(u_j^i \times u_i^j) & \text{if } u_j^i \times u_i^j \neq \mathbf{0} \\ \text{any unit length vector} \perp u_j^i & \text{otherwise} \end{cases}$$

$\alpha_j^i = a\tan_2(\|u_j^i \times u_i^j\|, -u_j^i \cdot u_i^j)$

Then all solutions to (7) are $R_j^i = \exp(\beta \hat{u}_j^i)\mathbf{H}_j^i$ $\beta \in [-\pi, \pi]$:arbitrary angle

Theorem 11(Orientation Localizability for 2-D)

Consider a relative sensing network with 2-D and with noiseless angle-of-arrival sensing. The following statements are equivalent:

(i) The sensing graph is connected

(ii) The network is orientation localizable

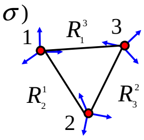
Orientation Localization with Noise[4]

Consider planer network

We can measure only $y_j^i = \mathrm{proj}((\angle p_j^i + n_j^i) - (\angle p_i^j + n_i^j) + \pi)$

n_j^i, n_i^j :Noise (Gaussian with zero mean and variance σ)

$\mathrm{proj}(\theta) = (\theta + \pi) \bmod 2\pi - \pi$



$R_2^1 R_3^2 R_1^3 = I$

Enforce cycle constraint and mitigate the noise

Assign a direction to each edge

If $i > j$ $i \to j$

Get directed graph $G_d = (V_s, E_d)$ Different from sensing graph!

Oriented edge $e = (j, i) \in E_d$ with $(i > j)$

Problem Settings[4]

Let

ψ_e : estimate of $\theta_e = \theta_i^j$

$\psi \in \mathbb{R}^m$: vector of angle estimates for all edges

y :measurement vector with $y_e = y_i^j$

Cycle error $\epsilon_l = \mathrm{proj}(\mathbf{1}_l \cdot \psi)$
$= \mathrm{proj}(\sum_{f \in l} \pm \psi_f) = 0$ Constraints (consistent estimates)

$\pm$: whether direction of the edge f is concordant with the direction of cycle l

Least-squares estimation problem

$$\min_{\psi} \|\psi - y\|^2$$

subject to $\mathrm{proj}(\mathbf{1}_l \cdot \psi) = 0$ for all $l \in \mathcal{L}(G_d)$

Estimation Algorithm[4]

Arbitrary network G_d

Subset of the cycle set: $\hat{\mathcal{L}} \subseteq \mathcal{L}(G_d)$ ψ_e : estimate of edge e

Cycle-distributed system

$$\psi_e(0) = y_e$$
$$\psi_e(t+1) = \psi_e(t) - \kappa \sum_{l \in \hat{\mathcal{L}}: e \in l} (\mathbf{1}_l \cdot \mathbf{e}_e)\mathrm{proj}(\mathbf{1}_l \cdot \psi_e(t)) \qquad 0 < \kappa << 1$$

$\mathbf{e}_i$: i th entry is 1, all the others are 0

Theorem 18 (Exponential convergence of iterative estimation algorithm)

The solution of cycle-distributed system with $\hat{\mathcal{L}} = \mathcal{L}_f$ converges exponentially fast with factor $\rho = (1-\kappa)^2$,with zero cycle error with $\kappa < 2/(1+\lambda_{\max}(F))$

$F = C_f C_f^T$ $\lambda_{\max}(F)$:Maximum eigenvalue of F

Proof of the Theorem[4]

$\dim(L_f) = r$

Fundamental cycles: $l_1, \ldots, l_r$

Cycle error vector $\epsilon = [\epsilon_{l_1}, \cdots, \epsilon_{l_r}]^T$

$$\psi(t+1) = \psi(t) - \kappa \sum_{l \in \mathcal{L}(G_d)} \mathbf{1}_l \epsilon_l(t)$$

For every loop $\alpha \in \mathcal{L}(G_d)$

$$\hat{\epsilon_\alpha}(t+1) = \hat{\epsilon_\alpha}(t) - \kappa \sum_{l \in \mathcal{L}(G_d)} (\mathbf{1}_\alpha \cdot \mathbf{1}_l)\epsilon_l(t) \qquad \hat{\epsilon_\alpha}(t) = (\mathbf{1}_\alpha \cdot \psi(t))$$

For all loop

$$\hat{\epsilon}(t+1) = \hat{\epsilon}(t) - \kappa C_f C_f^T \epsilon(t)$$
$$\epsilon(t+1) = \mathrm{proj}((I_r - \kappa F)\epsilon(t)) \qquad F = C_f C_f^T$$

Proof of the Theorem[4]

Consider linear system

$$x(t+1) = (I_r - \kappa F)x(t)$$

Lyapunov function candidate

$V(x) = x^T P x$ $P = I_r$

Define $Q = (2\kappa - \kappa^2)I_r > 0$ $A = I_r - \kappa F$

Discrete-time Lyapunov inequality $A^T P A - P \leq -Q$

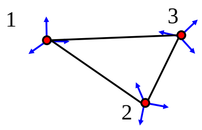
⇓

$\kappa < 2/(1+\lambda_{\max}(F))$ $P - Q = \rho I_r$ $\rho = (1-\kappa)^2$

3-D Frame Localization[4]

Lemma 19

Consider a network composed by three nodes in 3-D space with angle-of-arrival sensing. Pick any one of the three nodes as reference. If the sensing graph is the complete graph and the nodes are in generic position with generic orientations, then there are precisely tow feasible configurations for the three nodes and , therefore, the network is not orientation localizable.



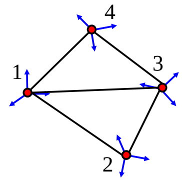
3-D Frame Localization[4]

Lemma 20

Consider a network composed by four nodes in the 3-D space with angle-of-arrival sensing.
If the sensing graph is connected and there are at least two independent loops, then the network is orientation localizable.

Lemma 21

A necessary condition for a network in the 3-D space with angle-of-arrival sensing to be orientation localizable is to have at least 4 nodes



Definition of Localization [5]

Definition 1 (Localized network) [5]

A network is said to be localized if there is a set of relative transformations g_{ij} such that, when the reference frame of the first node is fixed to g_1, the other absolute poses g_i are uniquely determined.
For any path *l* from node 1 to node *i*, we have $g_i = g_l \circ g_1$

Estimation of the Rotations[5]

Gradient of φ_R with respect to R_k

$$\mathrm{grad}_{R_k}\varphi_R = -R_k \sum_{k\to i} \log(R_j^T R_i \tilde{R}_{ki}^T) + \log(R_k^T R_i \tilde{R}_{ik})$$

Update procedure

$$R_k(l+1) = \exp_{R_k(l)}(-\varepsilon\, \mathrm{grad}_{R_k(l)})$$ ε : Step size

Initialization of the Rotations φ_R has multiple local minima

1. Set $R_1(0)=I, R_i(0)=\tilde{g}_{l_{1i}} R_1(0)$ Not distributed
2. New cost function $\varphi_R' = \frac{1}{2}\sum_{i\to j}\left\| R_j - R_i\tilde{R}_{ij}\right\|_F^2$

   Find $R_i(0)$ by minimizing φ_R' $\frac{\partial \varphi_R'}{\partial R_k} = \sum_{k\to i}(R_k R_i \tilde{R}_{ki}^T) + \sum_{i\to k}(R_k R_i \tilde{R}_{ik})$

φ_R' doesn't have local minima

Choice of the step size[5]

Minimization of quadratic form

$$\varphi = \frac{1}{2}\|My\|^2 \qquad M \in \Re^{q\times p} \qquad y \in \Re^p$$

- Quadratic cost function restricted to a line (direction of the grad.) is a parabola
- Maximum step-size is determined by maximum possible curvature of the parabola

  Related to maximum eigenvalue of $M^T M$

Gersgorin Discs and Gersgorin Theorem

Maximum eigenvalue can be substituted with the maximum of the absolute row sum of $M^T M$

Apply for minimizing φ_T ⇒ Computed in a distributed way

Calculation of the Gradient

Consider

$$\phi(R) = \frac{1}{2}\|R-Q\|_F^2 = \mathrm{trace}(I - R^T Q) \qquad \phi : SO(3) \to \mathbb{R}$$

SO(3) is submanifold of $\mathbb{R}^{3\times 3}$

Define $\bar{\phi}(\cdot) = \phi(\cdot)$ $\bar{\phi} : \mathbb{R}^{3\times 3} \to \mathbb{R}$

$$\mathrm{grad}_R \phi(R) = P_R \mathrm{grad}_R \bar{\phi}(R) \qquad (1)$$

Projection: $P_R : \mathbb{R}^{3\times 3} \to T_R SO(3)$ $P_R Z = R\mathrm{sk}(R^T Z)$

Tangent space: $T_R SO(3) = \{RX \in \mathbb{R}^{3\times 3} | X \in so(3)\}$

$$< \mathrm{grad}_R \bar{\phi}(R), Z >= D\bar{\phi}(R)[Z] \qquad (2)$$

$<\cdot,\cdot>$: Inner product $< Z_1, Z_2 >= \mathrm{trace}(Z_1^T Z_2)$ $Z_1, Z_2 \in \mathbb{R}^{3\times 3}$

Calculation of the Gradient

Directional derivative

$$D\bar{\phi}(R)[Z] = \lim_{t \to 0} \frac{\bar{\phi}(R + tZ) - \bar{\phi}(R)}{t} = -\mathrm{trace}(Z^T Q) \qquad (3)$$

From (2) and (3)

$$\mathrm{grad}_R \bar{\phi}(R) = -Q$$

$$\mathrm{grad}_R \phi(R) = P_R \mathrm{grad}_R \bar{\phi}(R) = P_R(-Q) = -R\,\mathrm{sk}(R^T Q)$$