Survey of Synchronization Part II: Synchronization on SO(3)

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FL11-14-1
4th, October, 2011

Related Works


In this seminar, we introduce results of [10] in detail, along with our future works.

Unit Quaternion

\[ \mathbf{q} = \cos(\theta/2), \quad \mathbf{q}_v = \xi \sin(\theta/2) \]  

\[ R = e^{i \mathbf{q} \cdot \mathbf{q}_v} \]

\[ R = I_3 + 2\mathbf{q}_v \times \mathbf{q} + 2\mathbf{q}_v^2 \]

\[ e^{i \phi} = I_3 + \xi \mathbf{q}_v + \xi^2 (1 - c_\theta) \]

Quaternion Product

\[ p \cdot q = [p_0 q_0 - q_1^T p_v, p_q q_0 + q_1^T p_v] \]

Kinematics

\[ \frac{dq_0}{dt} = -\frac{1}{2} (\omega^T) q_v, \quad \frac{dq_v}{dt} = \frac{1}{2} \omega q_v + \frac{1}{2} \omega^T q_v \]

Passivity of Unit Quaternion

\[ \frac{dR}{dt} = \omega^b R \]

The unit quaternion kinematics is passive from \( \omega^b \) to \( \mathbf{q}_v \).

Proof: \[ V_2 = 2(\mathbf{q}_0 - 1) \mathbf{q}_v^2 \geq 0 \]

\[ V_2 = 2(\mathbf{q}_0 - 1) \mathbf{q}_v^2 = -(\mathbf{q}_0 - 1) \mathbf{q}_v^2 \mathbf{q}_v + \mathbf{q}_v^2 \mathbf{q}_v \]

\[ V_2 = 2\mathbf{q}_v \mathbf{q}_v \]

\[ \mathbf{q}_v = \xi \sin(\theta/2), \quad V_2 = 2[1 - \cos(\theta/2)] = \xi (I_3 - R(\theta/2)) \]

\[ e^{i \theta} = e^{i \phi} e^{i \xi} \]

is passive from \( \omega^b \) to \( \xi \sin 6 \).

\[ V_1 = \frac{1}{2} \mathbf{q}_v (I_3 - R(\theta/2)) \]

Synchronization on SO(3)

Motivation

- Analysis of group behaviors such as schools of fish, flocking of birds, etc…
- Application to mobile sensor networks, formation control, etc…

Related Works

- Partial Stability: stability for the shortest axis
- Extensions of the results of [1,2] not for the shortest axis
- Extensions of the results of [1,2,3] 3D ver. of identical steered particles (all-to-all communication)

3D ver. of identical steered particles (all-to-all communication)

Appendix

Future works and applications.
**Preliminary: Incidence Matrix**

Graph Laplacian \( D: \) degree matrix  
\( I = D - A \)  
\( A: \) adjacency matrix  

Incidence Matrix \( B \in \mathbb{R}^{N \times N} \)  
for Oriented Bidirectional Graph  
with \( N \) Vertices and \( M \) Links  

\[
B_{ik} = \begin{cases} 
1 & \text{if } \text{link } k \text{ is incoming to vertex } i \\
-1 & \text{if } \text{link } k \text{ is outgoing from vertex } i \\
0 & \text{otherwise}
\end{cases}
\]

\( B = \begin{bmatrix} 
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix} \)

\( D = \begin{bmatrix} 
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix} \)

Oriented Bidirectional Graph

We consider only bidirectional graphs and Incidence Matrix \( B \)

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**Kinematics of the Attitude Error**

Relative Attitude for Link \( k \)

\[
\hat{\mathbf{R}}^k = R^k_1 R^k_1 \quad \text{if } k \in N^+ \quad \text{and } k \in N^- 
\]

Relative Angular Velocity for Link \( k \)

\[
\hat{\omega}^k = \omega^k_1 \quad \text{if } k \in N^+ \quad \text{and } k \in N^- 
\]

Relative Angular Velocities for All Links

\[
\hat{\omega} = B^T \hat{\omega}^k (1) \quad \text{if } k \in N^+ \\
\hat{\omega} = 0 \quad \text{if } k \in N^- 
\]

\[
\hat{\omega} = \omega^k_1 \quad \text{otherwise} 
\]

Kinematics of Relative Attitudes

\[
d\hat{\mathbf{R}}^k = (\mathbf{R}^k)^\Delta \hat{\mathbf{R}}^k 
\]

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**Dynamics of the Attitude**

\[
\dot{\mathbf{q}}^k = \frac{1}{2} \mathbf{q}^k_1 \mathbf{q}^k_2 \mathbf{R}^k - I_S + 2 \mathbf{q}^k_1 \dot{\mathbf{q}}^k_1 + 2 \dot{\mathbf{q}}^k_2 \mathbf{q}^k_2 \mathbf{R}^k 
\]

Kinematics of Relative Attitudes (Quaternion)

\[
\dot{\mathbf{q}}^k_1 = \frac{1}{2} \mathbf{q}^k_1 \mathbf{q}^k_2 \mathbf{R}^k - \dot{\mathbf{q}}^k_1 \mathbf{q}^k_2 \mathbf{R}^k + 2 \mathbf{q}^k_1 \dot{\mathbf{q}}^k_1 + 2 \dot{\mathbf{q}}^k_2 \mathbf{q}^k_2 \mathbf{R}^k 
\]

\[
\mathbf{d} \hat{\mathbf{R}}^k = (\mathbf{R}^k)^\Delta \mathbf{R}^k 
\]

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**Comment (1): Dynamics**

[11]: We use a similar approach for robot dynamics

[10]: Bidirectional Connected Graph

\[\text{Dynamics} \rightarrow \text{Passivity} \rightarrow \text{Passivity-based kinematic Control}\]

In [10],

"The work in [12] considered kinematic control of attitude synchronization. Since the agent kinematics are relative degree one, the attitude synchronization can be achieved with strongly connected graphs."

Perhaps, it is because of the choice of the Lyapunov function candidate (individual energy) based on [13] (discuss afterward)

We probably can include dynamics by simply adding the kinetic energy, but agents may have to use their own velocities (Future Work 1)

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**Passivity of Attitude Dynamics**

Angular Velocity Error

\[
\Delta \omega := \omega^k_1 - \omega^k_2
\]

Torque Input

\[
\tau_i = I_i \dot{\omega}^k + \dot{\mathbf{q}}^k_1 \mathbf{R}^k \dot{\mathbf{q}}^k_2 - f_i \Delta \omega_i + u_i \quad (5)
\]

Velocity Feedback

\[
\mathbf{H}_i : \dot{\mathbf{v}}_i = (\Delta \omega_i)^T f_i \Delta \omega_i + u_i \quad (6)
\]

\( u_i \) is strictly passive from \( \tau_i \) to \( \Delta \omega_i \)

Proof: \( \mathbf{v}_i = \frac{1}{2} (\Delta \omega_i)^T f_i \Delta \omega_i \geq \xi : \text{Kinetic Energy} \)

\[
\dot{\mathbf{v}}_i = (\Delta \omega_i)^T f_i \Delta \omega_i + u_i 
\]

\[
= - (\Delta \omega_i)^T (\Delta \omega_i)^T f_i \Delta \omega_i - (\Delta \omega_i)^T f_i \Delta \omega_i + (\Delta \omega_i)^T u_i 
\]

\[
= - (\Delta \omega_i)^T (\Delta \omega_i)^T f_i \Delta \omega_i + (\Delta \omega_i)^T u_i 
\]

negative definite

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**Attitude Synchronization Law**

Attitude Synchronization Law

\[
u_i = \sum_{j \in N^+_i} q^k_1 \sum_{j \in N^-_i} q^k_1 + q^k_2 \quad (7)
\]

\[
u_i = (B \otimes I_3) q_i \quad (8)
\]

Difference from [12]

(7): \( u_i = \sum_{j \in N^+_i} \xi_j \sin(\theta_j / 2) + \sum_{j \in N^-_i} \xi_j \sin(\theta_j / 2) \)

(8) uses the same control scheme as our works but...


[13]: N. Chopra and M. W. Spong. "Passivity-based Control of Multi-agent Systems." In Advances in Controle...
Attitude Synchronization

\[ 1_N \otimes \omega & \frac{d}{dt} \xi_i = \pi_i \left( I - I_i \otimes I_i \right) \quad \xi_i \in \mathbb{S}^1 \]

Passivity: \( V_{\xi_i} + \sum_{j \neq i} \xi_j \cdot \xi_j = 0 \)

**Theorem [10]**

If the graph is acyclic, then \( \xi_i \to 1 \) and \( \xi_i \to 0 \) as \( t \to \infty \)

Proof: \( V = V_{\xi_i} + \frac{1}{2} \xi_i^T F \xi_i \) : Sum of the Energies

\[ V = -\omega^T \xi_i + \xi_i^T F \xi_i = \left( 1_N \otimes \omega \right)^T \left( I - I_i \otimes I_i \right) \xi_i - \omega^T F \xi_i = -\xi_i^T F \xi_i \leq 0 \]

Comment (3): Proof Techniques

**Theorem [10]**

If the graph is acyclic, then \( \xi_i \to 0 \) and \( \xi_i \to 0 \) as \( t \to \infty \)

Proof: \( \dot{V} = -\xi_i^T F \xi_i \leq 0 \)

\( F \) : diagonal, \( F = \text{diag} \{ I_1, \ldots, I_n \} \)

Comment (5): \( \pi/2 \) Problem 2

\[ u_i = \sum_{j \neq i} \xi_j \sin \theta_{ij}/2 \]

Relative Information

\[ \begin{array}{c c}
120^\circ & 120^\circ \\
190^\circ & 190^\circ \\
\end{array} \]

Agents do not rotate

Indeed, we also find the same situation for not synchronization, but we have not found other situations yet in strongly connected graphs

If we overcome \( \pi/2 \) problem for directed, strongly connected graphs, we may be able to solve the same augment i.e. cyclic graphs are not good or almost globally stable except for some unstable equilibria

Adaptive Design for Desired Angular Velocities

So far, the desired velocity \( \omega^d(t) \) is available to each agent.

Only the leader possesses \( \omega^d(t) \) information and the remaining agents estimate it

In Hatanaka’s work [14], all agent know the same desired body velocity \( \omega^d(t) \), so this work can be a good reference

Assumption: \( \omega^d(t) = \sum_{j=1}^{n} \Phi^d(\tau_j) \beta_j \)

\( \beta_j \) : column vectors available only to the leader

That is, each agent does not know the axis of the desired angular velocity but know the value like an absolute one

Estimate of Unknown \( \beta_j \)

Estimate of the Desired Velocity

\[ \omega^d(t) = \sum_{j=1}^{n} \Phi^d(\tau_j) \beta_j \]

Passivity of the Closed-loop System

Estimate Error Dynamics
\[ \dot{\hat{\beta}}_i = K_i (\Phi(t) \otimes I_3) u_i \]
\[ (11) \]
(11), is passive from \((\Phi(t) \otimes I_3) u_i \) to \( \hat{\beta}_i \)
\[ V_\beta = 1/2 \hat{\beta}_i K_i \hat{\beta}_i \]

Error Dynamics
\[ I_1 \Delta \omega_1 + (\Delta \omega_1)^T I_1 \Delta \omega_1 = -f_1 \Delta \omega_1 + u_1 \]
\[ (12) \]
\[ I_1 \Delta \omega_1 + (\Delta \omega_1)^T I_1 \Delta \omega_1 = -f_1 \Delta \omega_1 + u_1 \]
\[ (13) \]
(12), (13) is strictly passive from \( u \) to \( \zeta \)
\[ V_u = 1/2 (\zeta_j^T \mathcal{F} \zeta_j^T + (\zeta_j^T F \zeta_j^T + (\zeta_j^T F \zeta_j^T \zeta_j^T u \]

Control schemes are all based on passivity properties

Summary of the Results of [9]

- Graph: bidirectional, acyclic, connected
- If the graph is cyclic, null space of \( B \otimes I \) is not 0 \( \times n \rightarrow 0 \)
- Control Law:
\[ u_i = \sum_{j \in R_i} \xi_j \sin(\theta_{ij}/2) - \sum_{j \notin R_i} \xi_j \sin(\theta_{ij}/2) \]
(7)
- No \( \pi/2 \) Problem: perhaps because Lyapunov function is based on relative attitude
- Proof Technique: bounded, Barbalat’s Lemma
- Adaptive Design for the Desired Angular Velocity
\[ \omega(t) = \sum_{i = 1}^{N} \phi(\theta(t)) + \sum_{i = 1}^{N} \phi(\theta(0)) \hat{\beta}_i = K_i (\Phi(t) \otimes I_3) u_i \]
only the leader knows the information
the approach is similar to Hatanaka’s work

Future Works

Future Works

1. Progress Report

Appendix
Barbalat’s Lemma

Let \( f(t) : \mathbb{R} \to \mathbb{R} \) be a uniformly continuous function on \((0, \infty)\). Suppose that \( \lim_{t \to \infty} f(s) ds \) exists and is finite. Then

\[ f(t) \to 0 \text{ as } t \to \infty. \]


Theorem

Suppose \( f(t, x) \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \), uniformly in \( t \), on \((0, \infty) \times \mathbb{R}^n\). Furthermore, suppose \( f(t, 0) \) is uniformly bounded for all \( t \geq 0 \). Let \( Y : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n \) be a continuously differentiable function such that

\[
W_1(x) \leq V(t, x) \leq W_2(x) \quad \forall t \geq 0, \forall x \in \mathbb{R}^n,
\]

\[
\dot{V}(x) = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} f(t, x) \leq -W(x) \quad \forall t \geq 0, \forall x \in \mathbb{R}^n,
\]

where \( W_1(x) \) and \( W_2(x) \) are continuous positive definite functions and \( W(x) \) is a continuous positive semidefinite function on \( \mathbb{R}^n \). Then, all solutions of \( \dot{x} = f(t, x) \) are bounded and satisfy

\[
W(x(t)) \to 0 \text{ as } t \to \infty.
\]

In the proof, \( V(t, x) = W_2(x) - \frac{1}{2} \int_0^t \dot{W}_2(x(s)) ds \).

Other Works [2,3,4,5]

Attitude Kinematics

\[
R(t) \dot{R}_k = \omega_k(t) \quad (14) \quad J_k \omega_k(t) = (J_k \omega_k(t))^T \omega_k(t) + \tau_k(t) \quad (15)
\]

Objective: Attitude Synchronization

\[
\lim_{t \to \infty} R_k^T R_j = I_k \quad k, j \in \{1, \cdots, N\}
\]

Distance between \( R_k \) and \( R_j \)

\[
d_{jk} := \sqrt{3 - \text{tr}(R_k^T R_j)} \quad \frac{1}{\sqrt{3 \sigma}} \left( \sqrt{3} - \text{tr}(R_k^T R_j) \right)
\]

Potential Energy [1]

\[
V = \frac{1}{2} \sum_{k \in \mathcal{N}_i} \sum_{j \in \mathcal{N}_i} \text{tr}(R_k^T R_j), \quad \sigma < 0 \quad \Rightarrow \quad \frac{1}{2} \text{tr}(I_3 - R_i)
\]

Because of the manifold structure of \( \text{SO}(3) \), unlike in Euclidean spaces, the cost function could have local minima

[2,3]: fixed, bidirectional, connected graphs

Utilize the consensus algorithm which guarantees global stability

Auxiliary Variables: \( Y_k \in \mathbb{R}^{3 \times 3} \)

\[
Y_k : \text{consensus} \Rightarrow R_k \text{ tracks the projection of } Y_k \text{ on } \text{SO}(3)
\]

Consensus Algorithm in Inertial Frame

\[
\dot{Y}_k = \beta \sum_{i \in \mathcal{N}_i} (Y_i - Y_k), \quad \beta > 0
\]

For the graph uniformly connected, consensus is achieved [18]

\[
\dot{X}_k = R_k^T Y_k
\]

\[
\dot{X}_k = \beta \sum_{i \in \mathcal{N}_i} (R_k^T R_i X_i - X_k) - \dot{\omega}_k \times X_k
\]


Sarlette, et al. [5]

Distance from \( Y_k \) to \( R_k \in \text{SO}(3) \) in \( \mathbb{R}^{3 \times 3} \)

\[
|Y_k - R_k|_F^2 = \text{tr}((Y_k - R_k)^T (Y_k - R_k)) = 3 + \text{tr}((Y_k^T Y_k) - 2tr(R_k^T Y_k))
\]

Tracking Algorithm of \( R_k \) to the Projection of \( Y_k \) on \( \text{SO}(3) \)

\[
R_k^T R_k = -\sigma (R_k^T Y_k - Y_k^T R_k)
\]

Thus, [5] proposes the following attitude synchronization law

\[
\tau_k = -\sigma (X_k - X_k^T)^T - \dot{\omega}_k \times \dot{X}_k \quad (16)
\]

\[
\dot{X}_k = \beta \sum_{j \in \mathcal{N}_i} (R_k^T R_i X_j - X_k) - \dot{\omega}_k \times X_k \quad (17)
\]

Theorem [5]

For the graph uniformly connected, attitude synchronization with \( \omega_k = 0 \) \( Y_k \) is almost globally stable for (14), (15) with controller (16), (17).