Optimal Inter-Vehicular Distance Control with Constraints in Vehicle Networks

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Background

Highway congestion

Highway congestion is imposing an intolerable burden on urban residents

Congestion occurs when vehicle's velocity variation propagates to following vehicles

It is difficult for human drivers to recognize tiny changing of the precede vehicle's velocity

Approaches

There are various approaches to improve congestion
They can be classified as
macro perspective and micro perspective

Macro perspective: On-ramp control, Transportation Network
Micro perspective: Vehicle Platoon Control

P. Varaiya, "Smart Cars on Smart Roads: Problems of Control", *IEEE Transactions on Automatic Control*, Vol. 38, No. 2, Feb. 1993

Review

Situation

Considering n vehicles in an one lane highway

Optimization Problem

$$\max_s \left\{ J_N = c^T q_N + \sum_{k=0}^{N-1} (\|Cq_k\|_1 - \|RS_k\|_1) \right\}$$

s.t. $\sum_{i=1}^{n} S_{i,k} = M \quad \dot{q}_k = A''_d\, q_k + B''_d\, S_k \quad q_k = [x_k, v_k]^T$

Proposition 4-1'

Consider the finite horizon optimization problem(*)' under the dynamics(**)'.
The Nth stage Optimal value of the DP iteration is (***)'.
The optimal control at time k, for k=0,1,…,N-1 is $S^*_{N-k} = (S^*_{1,N-k}, \cdots, S^*_{n,N-k})$
where

$$S^*_{i,N-k} = \begin{cases} M & i = \arg\max(h_k) \\ 0 & otherwise \end{cases}$$

New Approach

Situation

Considering n vehicles in an one lane highway

Optimization Problem

$$\max_s \left\{ J_N = c^T q_N + \sum_{k=0}^{N-1} (\|Cq_k\|_1 - \|RS_k\|_1) \right\}$$

s.t. $q_{k+1} = A''_d\, q_k + B''_d\, S_k \; q_k = [x_k, v_k]^T$

$Dx \geq 0$
Collision avoidance constraints

$$D = \begin{bmatrix} \ddots & & \\ & 1 & -1 \\ & & \ddots \end{bmatrix}$$

Objective

To analyze
- the active set of the constraints
- string stability
- (the difference of the optimal solution with constraints)

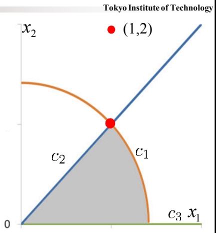
Active Set

Example

$$\min_x f(x) = (x_1-1)^2 + (x_2-2)^2$$

constraints $c_1(x) = x_1^2 + x_2^2 - 2 \leq 0$ strictly convex
$c_2(x) = -x_1 + x_2 \leq 0$
$c_3(x) = -x_2 \leq 0$

$$\Rightarrow \begin{cases} x_1^2 + x_2^2 - 2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \leftrightarrow \begin{cases} x_1 = \sqrt{2} \\ x_2 = \sqrt{2} \end{cases}$$



Meaning

Active set:
It determines which constraints influence the final result of optimization
In solving the linear programming problem,
the active set gives the hyperplanes that intersect at the solution point

If we know active set in advance,
we can make the constraints low-dimensional

Parameter Settings

Vehicle Dynamics

$q_{k+1} = A''_d\, q_k + B''_d\, S_k \quad q_k = [x_k, v_k]^T$

$$\Leftrightarrow q_{k+1} = \begin{bmatrix} I & tI \\ k_p t L'_g & k_v t L'_g + I \end{bmatrix} q_k - k_p t \begin{bmatrix} 0 \\ I \end{bmatrix} S_k$$

$L'_g = -L_g$: Communication graph
t : Sampling time

$x_{k,m}$: k step, mth vehicle

Communication graph

Considering the simplest graph

$$L'_g = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ & & \ddots & \ddots \end{bmatrix}$$

Feedback the precede vehicle information

Input(Inter-vehicular distance)

Time-invariant constant inter-vehicular distance

$S_k = \mathbf{M} \quad \mathbf{M} = [0, M, M, \cdots]^T$ Considering the case of one input

Constraints analysis

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Deformation the collision avoidance constraints

$$D = \begin{bmatrix} \ddots & & & \\ & 1 & -1 & \\ & & & \ddots \end{bmatrix}$$

$Dx \geq 0$

$\Leftrightarrow Dx_0 \geq 0, Dx_1 \geq 0, \cdots, Dx_N \geq 0$

$$\begin{cases} \begin{bmatrix} x_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} I & tI \\ k_p t L'_g & k_v t L'_g + I \end{bmatrix} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} - k_p t \begin{bmatrix} 0 \\ I \end{bmatrix} \mathbf{M} = \begin{bmatrix} x_0 + t v_0 \\ k_p t L'_g x_0 + k_v t L'_g v_0 - k_p t \mathbf{M} \end{bmatrix} \\ \begin{bmatrix} x_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} I & tI \\ k_p t L'_g & k_v t L'_g + I \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \end{bmatrix} - k_p t \begin{bmatrix} 0 \\ I \end{bmatrix} \mathbf{M} \\ \quad = \begin{bmatrix} x_0 + 2t v_0 + t^2 (k_p L'_g x_0 + k_v L'_g v_0 - k_p \mathbf{M}) \\ k_p t L'_g (x_0 + t v_0) + k_v L'_g (k_p t L'_g x_0 + k_v t L'_g v_0 - k_p t \mathbf{M}) - k_p t \mathbf{M} \end{bmatrix} \\ \vdots \end{cases}$$

$\Leftrightarrow \underbrace{Dx_0 \geq 0, Dx_1 \geq 0,}_{\text{Assumption}} \underbrace{Dx_2(M) \geq 0, \cdots, Dx_N(M) \geq 0}_{\text{Constraints}}$

Constraints analysis

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Analysis

$$Dx_2(M) = Dx_0 + 2tDv_0 + t^2 D(k_p L'_g x_0 + k_v L'_g v_0 - k_p \mathbf{M}) \qquad L'_g = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ & & \ddots & \ddots \end{bmatrix}$$

$D\mathbf{M} = [M \quad 0 \quad \cdots]^T \qquad \mathbf{M} = [0, M, M, \cdots]^T$

$$= \begin{bmatrix} x_{0,1} - x_{0,2} \\ x_{0,2} - x_{0,3} \\ \vdots \end{bmatrix} + 2t \begin{bmatrix} v_{0,1} - v_{0,2} \\ v_{0,2} - v_{0,3} \\ \vdots \end{bmatrix} + t^2 k_p \begin{bmatrix} -x_{0,1} + x_{0,2} \\ x_{0,1} + x_{0,3} \\ \vdots \end{bmatrix} + t^2 k_v \begin{bmatrix} -v_{0,1} + v_{0,2} \\ v_{0,1} + v_{0,3} \\ \vdots \end{bmatrix} - t^2 k_p \begin{bmatrix} M \\ 0 \\ 0 \end{bmatrix}$$

⇨ Constraints

between 1,2 vehicles: $(1 - t^2 k_p)(x_{0,1} - x_{0,2}) + (2t - t^2 k_v)(v_{0,1} - v_{0,2}) + t^2 k_p M \geq 0$

others: $t^2 k_p x_{0,m-1} + x_{0,m} + (k_p t^2 - 1) x_{0,m+1} + t^2 k_v v_{0,m-1} + v_{0,m} + (k_v t^2 - 1) v_{0,m+1} \geq 0$

Only constraint between 1,2 vehicles bound input M
Other collision conditions depend on the initial condition, not input M

Constraints analysis

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$$v_2 = k_p t L'_g (x_0 + t v_0) + k_v L'_g (k_p t L'_g x_0 + k_v t L'_g v_0 - k_p t \mathbf{M}) - k_p t \mathbf{M} \qquad L'_g = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ & & \ddots & \ddots \end{bmatrix}$$

$L'_g \mathbf{M} = \mathbf{0}$

M doesn't remain the later step, just influence 1 step $\quad \mathbf{M} = [0, M, M, \cdots]^T$

⇨ $Dx \geq 0$

$\Leftrightarrow x_{2,1} - x_{2,2}(M) \geq 0, \cdots, x_{N,1} - x_{N,2}(M) \geq 0$

Proposition

In precede following communication,
we can't control the collision avoidance constraints from M

Active set

$$J_N = c^T q_N + \sum_{k=0}^{N-1} (\|Cq_k\|_1 - \|RS_k\|_1) = c^T q_N + \sum_{k=0}^{N-1} (c^T q_k - r^T \mathbf{M})$$

⇨
$$\max_M \left\{ c^T q_N + \sum_{k=0}^{N-1} (c^T q_k - r^T \mathbf{M}) \right\}$$
s.t. $q_{k+1} = A''_d q_k + B''_d \mathbf{M}$
$x_{2,1} - x_{2,2}(M) \geq 0, \cdots, x_{N,1} - x_{N,2}(M) \geq 0$

Constraints analysis

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$$A''_d = \begin{bmatrix} I & tI \\ k_p t L'_g & k_v t L'_g + I \end{bmatrix} \qquad B''_d = -k_p t \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\begin{aligned} q_{k+1} &= A''_d q_k + B''_d \mathbf{M} \\ &= A''^2_d q_{k-1} + A_d B''_d \mathbf{M} + B''_d \mathbf{M} \\ &= A''^2_d q_{k-1} - k_p t \begin{bmatrix} tI \\ I \end{bmatrix} \mathbf{M} + B''_d \mathbf{M} \leftarrow L'_g \mathbf{M} = \mathbf{0} \\ &= A''^{k+1}_d q_0 + \underbrace{\sum_{j=0}^{k} j B'''_d \mathbf{M} + \sum_{j=0}^{k+1} B''_d \mathbf{M}}_{\text{Independent from } A} \leftarrow B'''_d = -k_p t^2 \begin{bmatrix} I \\ 0 \end{bmatrix} \end{aligned}$$

$$\max_M \left\{ c^T q_N + \sum_{k=0}^{N-1} (c^T q_k - r^T \mathbf{M}) \right\} \qquad \min_{M'} c'^T \mathbf{M}$$

The optimal solution of the linear programming is
the hardest constraints of $x_{2,1} - x_{2,2}(M) \geq 0, \cdots, x_{N,1} - x_{N,2}(M) \geq 0$

The hardest constraints is active set

⇨ be able to draw an initial condition-active set map

Active Set Map

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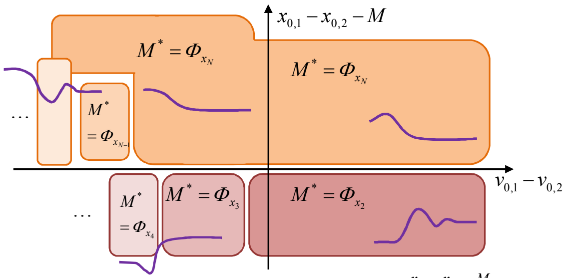
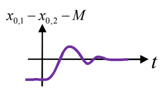
System is stable, there is no disturbance, assume $Dx_0 \geq 0, Dx_1 \geq 0$





Φ_{x_2}: Solution of $x_{2,1} - x_{2,2}(M) = 0$

Summary

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- Optimization problem with collision avoidance constraints
- Active set analysis
- Problem of precede following

Future works

- Analyzing the relationship between graph laplacian and constraints
- Analyzing the relationship between gain and string stability

Appendix

Consideration about Graph Laplacian

Constant inter-vehicular distance can be expressed as

$S_k = ML'_g J \qquad J = [1, 2, \cdots, n]^T$

$Dx \geq 0$

$x_k = x_{k-1} + tv_{k-1} = x_{k-2} + 2tv_{k-2} + t^2(k_p L'_g x_{k-2} + k_v L'_g v_{k-2} - k_p ML'_g J)$

⇨ $Dx_k = Dx_{k-2} + 2tDv_{k-2} + t^2 D(k_p L'_g x_{k-2} + k_v L'_g v_{k-2} - k_p ML'_g J)$

If we'd like to make the constraints including M, $DL'_g J \neq \mathbf{0}$

$DL'_g J \neq \mathbf{0}$

⇔ $\begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & & \ddots \end{bmatrix} L'_g J \neq \mathbf{0}$ ⇔ $L'_g \begin{bmatrix} 1 \\ 2 \\ \vdots \end{bmatrix} \neq k \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix}$

⇨ Each vehicle have a deferent communication graph from the one of precede vehicle

Consideration about Graph Laplacian

Relative communication structure

× $\begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ \vdots \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ \vdots \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$

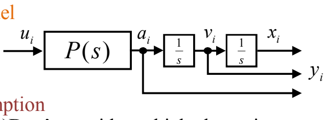
○ $\begin{bmatrix} 1 & -1 & & \\ 1 & 1 & -2 & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ \vdots \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ \vdots \end{bmatrix} \qquad \begin{bmatrix} 1 & -n & 1 & \cdots \\ 1 & 1 & -n & \cdots \\ & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{1}{2}n^2 - \frac{3}{2}n \\ \frac{1}{2}n^2 - \frac{5}{2}n \\ \vdots \end{bmatrix}$

⇨ It would be better for vehicles to feedback the absolute information(reference) such as a leader

$\begin{bmatrix} 1 & -1 & & & \\ 1 & 1 & -2 & & \\ 1 & & 1 & -2 & \\ \vdots & & & & \ddots \end{bmatrix}$

Modeling

Vehicle model



Assumption

(A4)Don't consider vehicle dynamics

⇔ $P(s) = 1$, $a_i = u_i$

⇨ $\begin{bmatrix} \dot{x}_i \\ \dot{v}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i$ ⇔ $\dot{q}_i = Aq_i + Bu_i \quad q_i = [x_i, v_i]^T$

(A1)⇔$q_i \geq 0$

Platoon model

(A2)⇔ $\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$ ⇔ $\dot{q} = A'q + B'u \quad q = [x, v]^T$

$q \geq 0$

$x = [x_1, \cdots, x_n]^T \quad u = [u_1, \cdots, u_n]^T$

$v = [v_1, \cdots, v_n]^T$

Modeling

Control input

$u_i = K_p(s)(e_{x,i} - S_i) + K_v(s)e_{v,i} \qquad e_{x,i} = \sum_{k=1}^{n} \delta_k x_k - \left(\sum_{k=1}^{n} \delta_k\right) x_i \quad e_{v,i} = \sum_{k=1}^{n} \delta_k v_k - \left(\sum_{k=1}^{n} \delta_k\right) v_i$

Assumption

(A5) K_p, K_v are constant gain which satisfy (A1)

There exists some L_g such that

$u = K_p(e_x - S) + K_v e_v$

$e_x = L_g x \quad e_v = L_g v$

ex) $e_{x,1} = (-x_1 + x_3)$, $e_{x,2} = (x_1 - x_2)$, $e_{x,3} = (x_1 - x_3) + (x_2 - x_3)$ ⇨ $e_x = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} x$

Assumption

(A6) L_g is time invariant

Platoon model

$\dot{q} = A'q + B'u$

⇔ $\dot{q} = \begin{bmatrix} 0 & I \\ K_p L_g & K_v L_g \end{bmatrix} q - K_p \begin{bmatrix} 0 \\ I \end{bmatrix} S$ ⇔ $\dot{q} = A''q - B''S$

$q \geq 0$

Platoon model

$\dot{q} = A''q + B''S \quad q = [x^T, v^T]^T \geq 0 \quad S \leq 0$

String Stability

Definition[1]

Consider a string of N dynamic systems .the error signals $e(t)$ *depends on the disturbances* $d(t)$ *in the following manner:*

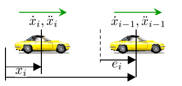
$e(t) = H_{e,d}(s)d(t) \qquad e, d \in R^n \qquad H_{n,d}(s) : R^N \rightarrow R^N \qquad (*)$

The system (*) is L_2 *string stable if given any* $\varepsilon > 0$ *there exist a* $\delta > 0$ *such that*

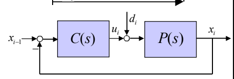
$\|d(\cdot)\|_2 < \delta \Rightarrow \|e(\cdot)\|_2 < \varepsilon$

Assumption

- LTI SISO plant/controller
- Each loop has relative degree
- Homogeneous loop



Deformation



$\|d(\cdot)\|_2 < \delta \Rightarrow \|e(\cdot)\|_2 < \varepsilon$

$\|G(s)\|_\infty = \sup_{d(t)} \frac{\|e(t)\|_2}{\|d(t)\|_2}$ [2]

$\|H_{e,d}(s)\|_\infty < \gamma \qquad \gamma = \frac{\varepsilon}{\delta}$

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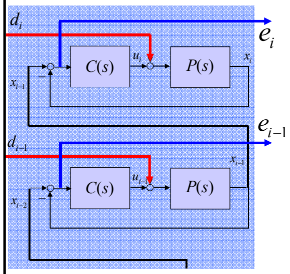
x_i : ith vehicle's position

u_i : input

d_i: disturbance

e_i : error

String Stability



From [3],

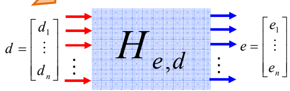
If $\left\|\frac{e_i(s)}{e_{i-1}(s)}\right\|_\infty < 1$, then $\exists\gamma > 0$ such that

$\left\|H_{e,d}(s)\right\|_\infty < \gamma, \forall N$

sufficient condition:

$$\left\|\frac{e_i(s)}{e_{i-1}(s)}\right\|_\infty < 1$$

The perturbation doesn't propagate to following vehicles



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