Equilibrium assignment of Transportation Network

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Introduction

Highway congestion

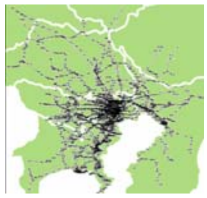
Highway congestion is imposing an intolerable burden on may urban residents

Congestion occurs when the demand for travel exceeds highway capacity

Action

There are some approaches to reducing congestion such that vehicle control, raising tolls or taxes etc..

All approaches boil down to the controlling highway capacity



P. Varaiya, "Smart Cars on Smart Roads: Problems of Control", *IEEE Transactions on Automatic Control*, Vol. 38, No. 2, Feb. 1993

Beginning

G. Como, K. Savla, D. Acemoglu, M. A. Dahleh and E.Frazzoli: "On Robustness Analysis of Large-scale Transportation Networks," *International Symposium on Mathematical Theory of Networks and Systems, Budapest, Hungary*, 2399-2406, 2010

Introduction

・Robustness analysis of transportation network
・Deriving upper and lower bound of stability margin
・Robustness price of anarchy

Ketan Savla
Massachusetts Institute of Techinology

Emilio Frazzoli
Massachusetts Institute of Techinology

Outline

・Introduction

・Problem Formulation
  Transportation network
  Route choice behavior

・Equilibrium Assignment Problem
  User equilibrium with fixed demand
  System optimum

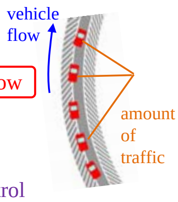
・Conclusion

Congestion Avoid

Highway capacity

To improve Highway capacity,

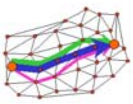
assign amount of traffic | increase vehicle flow



Traffic assignment problem

Transportation network

Highway is considered as a kind of network
→ User equilibrium assignment
  System optimum assignment



Vehicle control

Longitudinal control

Each vehicle should keep a fixed distance of the precede
→String stability

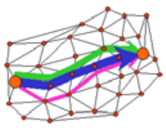
Lateral control

Vehicles seek to change a less crowed lane

Traffic Assignment Problem

Transportation network

Transportation network is a network of roads, streets and considered as a type of directed graph



Traffic assignment problem

・Concerning the traffic flow between origins and destinations in transportation networks
・To enable vehicles to reach destination in the shortest possible time using the maximum roadway capacity

Wardrop equilibrium

User equilibrium assignment | System optimum assignment

Problem Formulation

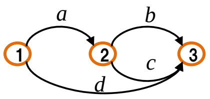
Notations

Directed graph

Transportation network is described by a Directed graph $G[N, A]$

N:Set of nodes
→Expressing City, Crossing etc..

A:Set of links
→Expressing Road



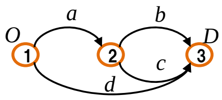
$N: 1,2,3$
$A: a,b,c,d$

Origin-Destination(OD)

$R \in N$:Set of origin nodes

$S \in N$:Set of destination nodes

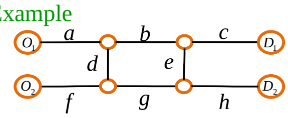
$K_{rs}, r \in R,\ s \in S$
:Set of simple route



$R: 1$
$S: 3$
$K_{13}: [a,b],[a,c],[d]$

Problem Formulation

Example



$R: O_1, O_2$
$S: D_1, D_2$
$A: a,b,c,d,e,f,g,h$

$K_{O_1D_1}: k_1 = [a,b,c], k_2 = [a,d,g,e,c]$ $K_{O_1D_2}: k_1 = [a,d,g,h], k_2 = [a,b,e,h]$

$K_{O_2D_1}: k_1 = [f,d,b,c], k_2 = [f,g,e,c]$ $K_{O_2D_2}: k_1 = [f,g,h], k_2 = [f,d,b,e,h]$

Traffic flow

Flow [pcu/min]

f_k^{rs} :Traffic flow of route k
x_a :Traffic flow of link a
pcu: passenger car unit

Cost [min]

t_a :Cost on link a
c_k^{rs} :Time taken to traverse route k

Example

$x_b = f_1^{O_1D_1} + f_2^{O_1D_2} + f_1^{O_2D_1} + f_2^{O_2D_2}$ $c_2^{12} = t_a + t_d + t_g + t_e + t_c$

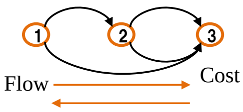
Problem Formulation

Notations

Flow

$x_a, a \in A$:Traffic flow of link a

$f_k^{rs}, r \in R, s \in S, k \in K_{rs}$
:Traffic flow of route k



Cost

$t_a = t_a(x_a)$:Cost on link a
→Expressing Toll, Time etc..

c_k^{rs} :Time taken to traverse route k

Relational expression

$$x_a = \sum_{rs}\sum_{k} \delta_{a,k}^{rs} f_k^{rs} \qquad c_k^{rs} = \sum_{a \in A} \delta_{a,k}^{rs} t_a(x_a)$$

$$\sum_k f_k^{rs} = Q_{rs} \qquad t_a(x_a) \ge 0$$

$$f_k^{rs} \ge 0$$

Q_{rs} :OD traffic flow

$\delta_{a,k}^{rs}$:Identification variable

Route Choice Behavior

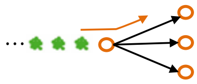
User model

Rational model

- Each user non-cooperatively seeks to minimize his cost of transportation
- Each user has a full information about all available routes

→All or nothing network loading

Stochastic rational model



- It's not always true that
  - -each user non-cooperatively seeks to minimize his cost of transportation
  - -each user has a full information about all available routes
- There is an error in observation data

→Stochastic network loading

- Logit model
- Probit model

Outline

- Introduction
- Problem Formulation
  - Transportation network
  - Route choice behavior
- Equilibrium Assignment Problem
  - User equilibrium with fixed demand
  - System optimum
- Conclusion

User Equilibrium

Wardrop equilibrium

Wardrop's first principle

The journey times in all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route

Wardrop's second principle

At equilibrium the average journey time is minimum

Assumption

- Each user non-cooperatively seeks to minimize his cost of transportation
- Each user has a full information about all available routes

User equilibrium

No user can minimize his journey time by unilaterally changing his or her route

User Equilibrium with Fixed Demand

User equilibrium with fixed demand

Assumption

・Fixed demand

・Each user chooses his route by Wardrop's first principle

Formulation of User equilibrium

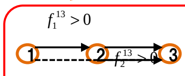
If $f_k^{rs} > 0,\ c_k^{rs} = c_{rs}\quad \forall k \in K_{rs}, \forall r \in R, \forall s \in S$

If $f_k^{rs} = 0,\ c_k^{rs} \ge c_{rs}\quad \forall k \in K_{rs}, \forall r \in R, \forall s \in S$

$$c_k^{rs} = \sum_{a \in A} \delta_{a,k}^{rs} t_a(x_a)$$

$$x_a = \sum_k \sum_{rs} \delta_{a,k}^{rs} f_k^{rs}$$

$f_1^{13} > 0$



$f_3^{13} = 0$

$c_3^{13} \ge c_1^{13} = c_2^{13} = c_{13}$

Constraint

$$\sum_k f_k^{rs} = Q_{rs}$$

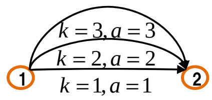
$f_k^{rs} \ge 0$

c_{rs} :Shortest time taken to traverse set of route rs

User Equilibrium with Fixed Demand

Example

Problem



Cost function

$t_1 = 5 + 0.1 f_1$

$t_2 = 10 + 0.025 f_2$

$t_3 = 15 + 0.025 f_3$

$Q_{12} = 200$

f [pcu/min]

pcu: passenger car unit

t [min]

Optimal problem

Want to get f_k^{rs} to satisfy User equilibrium

Find minimum c_k^{rs}

Constraint $c_k^{rs} = \sum_{a \in A} \delta_{a,k}^{rs} t_a(x_a)\quad x_a = \sum_k \sum_{rs} \delta_{a,k}^{rs} f_k^{rs}$

$\sum_k f_k^{rs} = Q_{rs}\quad f_k^{rs} \ge 0$

User Equilibrium with Fixed Demand

Mathematical optimization problem

Nonlinear optimal problem with constraints

$$\min Z_p = \sum_{a \in A} \int_0^{x_a} t_a(w)\,dw$$

Constraint

$\sum_k f_k^{rs} - Q_{rs} = 0\qquad x_a = \sum_k \sum_{rs} \delta_{a,k}^{rs} f_k^{rs}$

$f_k^{rs} \ge 0\qquad x_a \ge 0$

Karush-Kuhn-Tucker condition

If $f_k^{rs} > 0,\ \dfrac{\partial L(f^*, \lambda^*)}{\partial f_k^{rs}} = 0$ If $f_k^{rs} = 0,\ \dfrac{\partial L(f^*, \lambda^*)}{\partial f_k^{rs}} \ge 0\quad \dfrac{\partial L(f^*, \lambda^*)}{\partial \lambda_{rs}} = 0$

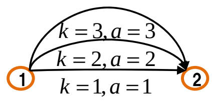
Lagrangian function

$$L(f, \lambda) = Z_p(f) - \sum_{rs} \lambda_{rs} \{ \sum_k f_k^{rs} - Q_{rs} \}\quad f_k^{rs} \ge 0$$

User Equilibrium with Fixed Demand

Example

Problem



Cost function

$t_1 = 5 + 0.1 f_1$

$t_2 = 10 + 0.025 f_2$

$t_3 = 15 + 0.025 f_3$

$Q_{12} = 200$

f [pcu/min]

pcu: passenger car unit

t [min]

Optimal problem

$$\min Z_p = 5 f_1 + 0.05 f_1^2 + 10 f_2 + 0.0125 f_2^2 + 15 f_3 + 0.0125 f_3^2$$

$$s.t.\ \sum_{k=1}^{3} f_k = 200\qquad f_k \ge 0 (k = 1 \sim 3)$$

Karush-Kuhn-Tucker condition

$$L(f, \lambda) = 5 f_1 + 0.05 f_1^2 + 10 f_2 + 0.0125 f_2^2 + 15 f_3 + 0.0125 f_3^2$$

$$- \lambda (\sum_{k=1}^{3} f_k - 200)\qquad f_k \ge 0 (k = 1 \sim 3)$$

User Equilibrium with Fixed Demand

$t_1 : 5 + 0.1 f_1 = \lambda (f_1 \ge 0)\ \ or\ \ 5 + 0.1 f_1 \ge \lambda (f_1 = 0)$

$t_2 : 10 + 0.025 f_2 = \lambda (f_2 \ge 0)\ \ or\ \ 10 + 0.025 f_2 \ge \lambda (f_2 = 0)$

$t_3 : 15 + 0.025 f_3 = \lambda (f_3 \ge 0)\ \ or\ \ 15 + 0.025 f_3 \ge \lambda (f_3 = 0)$

$\sum_{k=1}^{3} f_k - 200 = 0$

Search optimal solution

Case1: User chooses only Route 1

$f_1 = 200,\ f_2 = 0,\ f_3 = 0, t_1 = 25, t_2 = 10, t_3 = 15$

$\to Z_p = 3000,\ \lambda = t_1$

$\lambda > t_2, t_3$ Not optimal!

Case2: User chooses Route 1,2

$t_1 = t_2 = \lambda,\ f_3 = 0,\ \sum_{k=1}^{3} f_k - 200 = 0$

$\to f_1 = 80,\ f_2 = 120,\ f_3 = 0, t_1 = 13, t_2 = 13, t_3 = 15,$

$Z_p = 2100,\ \lambda = t_1, t_2$

$\lambda < t_3$ Optimal!

System Optimum

Wardrop equilibrium

Wardrop's first principle

The journey times in all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route.

Wardrop's second principle

At equilibrium the average journey time is minimum

Assumption

・Routes of all vehicles are controlled by the system

・Routing is based on maximum utilization of resources and minimum travel time

System optimum

All routes between a given OD pair have the same marginal travel time

System Optimum

Mathematical optimization problem

Nonlinear optimal problem with constraints

$$\min Z_s = \sum_{a\in A} x_a t_a(x_a)$$

Constraint

$$\sum_k f_k^{rs} - Q_{rs} = 0 \qquad x_a = \sum_k \sum_{rs} \delta_{a,k}^{rs} f_k^{rs}$$

$$f_k^{rs} \ge 0 \qquad x_a \ge 0$$

$$\min Z_s = \sum_{a\in A} x_a t_a(x_a) = \sum_{a\in A} \int_0^{x_a} [\frac{d\{w t_a(w)\}}{dw}] dw$$

Product rule

$$= \sum_{a\in A} \int_0^{x_a} [t_a(w) + w\frac{d\{t_a(w)\}}{dw}] dw$$

→This optimal problem can solve the same way as User equilibrium with fixed demand

User Equilibrium⇔System Optimum

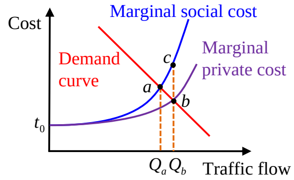
Aspect

User equilibrium

→Marginal private cost

System optimum

→Marginal social cost



At the equilibrium point,

Each user's cost is b under user equilibrium, and social cost is c

Social cost is a under social optimum

Social optimum is better than user equilibrium from the social cost

User equilibrium $\min Z_p = \sum_{a\in A} \int_0^{x_a} t_a(w) dw$

System optimum $\min Z_s = \sum_{a\in A} \int_0^{x_a} [t_a(w) + w\frac{d\{t_a(w)\}}{dw}] dw$

A kind of congestion toll

Conclusion

Summary

- Problem formulation
  - Transportation network
  - Route choice behavior
- Equilibrium assignment problem
  - -User equilibrium with fixed demand
  - -System optimum

Future works

- Equilibrium assignment problem
  - -Stochastic user equilibrium
  - -User equilibrium with variable demand
- Solution method of user equilibrium

Appendix

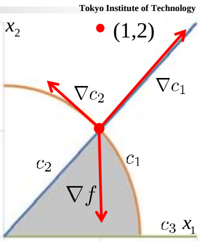
Karush-Kuhn-Tucker Conditions

Example $\min f(\mathbf{x}) = (x_1-1)^2 + (x_2-2)^2$

constraints $c_1(\mathbf{x}) = x_1^2 + x_2^2 - 2 \le 0$ strictly convex

$c_2(\mathbf{x}) = -x_1 + x_2 \le 0$

$c_3(\mathbf{x}) = -x_2 \le 0$



$$\nabla f(\mathbf{x}^*) - u_1^* \nabla c_1(\mathbf{x}^*) - u_2^* \nabla c_2(\mathbf{x}^*) = 0$$

$$\begin{pmatrix} 2(x_1^*-1) \\ 2(x_2^*-2) \end{pmatrix} + u_1^* \begin{pmatrix} 2x_1^* \\ 2x_2^* \end{pmatrix} + u_2^* \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$$

Karush-Kuhn-Tucker Conditions

$$L(x,\lambda) = \sum_{r\in S} U_r(x_r) - \sum_{r:l\in r} \lambda_r (x_r - c_r) \qquad \lambda_r \ge 0$$

the optimal value x^* and constraint's grad is Linear independence, then there exists constants λ_r such that

$$\partial L(x^*)/\partial x = 0 \qquad \lambda_r(x_r - c_r) = 0$$

Karush-Kuhn-Tucker Conditions(General Form)

$$\max_x f(x) \quad \text{s.t. } g_i(x) \le 0, \; h_j(x) = 0$$

the Lagrangian

$$L(x,\lambda,\mu) = f(x) - \sum_i \lambda_i g_i(x) + \sum_j \mu_j h_j(x)$$

Suppose there exists constants $\lambda_i \ge 0$ and μ_j such that

$$\frac{\partial f}{\partial x_k}(x^*) - \sum_i \lambda_i \frac{\partial g_i}{\partial x_k}(x^*) + \sum_j \mu_j \frac{\partial h_j}{\partial x_k}(x^*) = 0 \;, \forall k$$

$$\lambda_i g_i(x^*) = 0 \;, \forall i$$

the first-order necessary condition for optimality

Uniqueness of User Equilibrium with Fixed Demand

Nonlinear optimal problem with constraints

$$\min Z_p = \sum_{a \in A} \int_0^{x_a} t_a(w)dw$$

Constraint

$$\sum_k f_k^{rs} - Q_{rs} = 0 \quad x_a = \sum_k \sum_{rs} \delta_{a,k}^{rs} f_k^{rs} \quad f_k^{rs} \geq 0 \quad x_a \geq 0$$

$$\frac{\partial Z_p}{\partial x_a} = t_a(x_a)$$

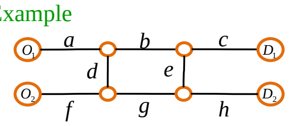
$$\frac{\partial^2 Z_p}{\partial x_a x_b} = \begin{cases} \dfrac{dt_a(x_a)}{dx_a} > 0 \, (a=b) \\ 0 \, (a \neq b) \end{cases} \quad \therefore \nabla^2 Z_p = \begin{bmatrix} \dfrac{dt_1(x_1)}{dx_1} & \cdots & 0 \\ \vdots & \dfrac{dt_a(x_a)}{dx_a} & \vdots \\ 0 & \cdots & \dfrac{dt_n(x_n)}{dx_n} \end{bmatrix}$$

Cost function is monotonic increase $\dfrac{dt_a(x_a)}{dx_a} > 0$

$\mathbf{h} \neq \mathbf{0}$ s.t. ${}^t\mathbf{h}(\nabla^2 Z_p)\mathbf{h} = \sum_{a \in A} h_a^2 \dfrac{dt_a(x_a)}{dx_a} > 0$ $\quad Z_p$ is convex function

Problem Formulation

Example



$R: O_1, O_2$

$S: D_1, D_2$

$A: a,b,c,d,e,f,g,h$

$K_{O_1D_1}: k_1 = [a,b,c], k_2 = [a,d,g,e,c] \quad K_{O_1D_2}: k_1 = [a,d,g,h], k_2 = [a,b,e,h]$

$K_{O_2D_1}: k_1 = [f,d,b,c], k_2 = [f,g,e,c] \quad K_{O_2D_2}: k_1 = [f,g,h], k_2 = [f,d,b,e,h]$

$$x_a = \sum_{rs} \sum_k \delta_{a,k}^{rs} f_k^{rs} = f_1^{O_1D_1} + f_2^{O_1D_1} + f_1^{O_1D_2} + f_2^{O_1D_2}$$

$$c_2^{12} = \sum_{a \in A} \delta_{a,2}^{12} t_a(x_a) = t_a(x_a) + t_d(x_d) + t_g(x_g) + t_e(x_e) + t_c(x_c)$$

x_a :Traffic flow of link a

f_k^{rs} :Traffic flow of route k

$t_a(x_a)$:Cost on link a

c_k^{rs} :Time of route k

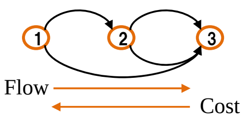
$\delta_{a,k}^{rs}$:Identification variable

Problem Formulation

Flow

$x_a, a \in A$:Traffic flow of link a

$f_k^{rs}, r \in R, s \in S, k \in K_{rs}$:Traffic flow of route k



Cost

$t_a = t_a(x_a)$:Cost on link a

→Expressing Toll, Time etc..

c_k^{rs} :Time taken to traverse route k

Relational expression

$$x_a = \sum_{rs} \sum_k \delta_{a,k}^{rs} f_k^{rs} \quad c_k^{rs} = \sum_{a \in A} \delta_{a,k}^{rs} t_a(x_a)$$

$$\sum_k f_k^{rs} = Q_{rs} \quad t_a(x_a) \geq 0$$

$$f_k^{rs} \geq 0$$

Q_{rs} :OD traffic flow

$\delta_{a,k}^{rs}$:Identification variable

User Equilibrium with Fixed Demand

Mathematical optimization problem

Nonlinear optimal problem with constraints

$$\min Z_p = \sum_{a \in A} \int_0^{x_a} t_a(w)dw$$

Constraint

$$\sum_k f_k^{rs} - Q_{rs} = 0 \quad x_a = \sum_k \sum_{rs} \delta_{a,k}^{rs} f_k^{rs}$$

$$f_k^{rs} \geq 0 \quad x_a \geq 0$$

Karush-Kuhn-Tucker condition

If $f_k^{rs} > 0, \ \dfrac{\partial L(f^*, \lambda^*)}{\partial f_k^{rs}} = 0$ If $f_k^{rs} = 0, \ \dfrac{\partial L(f^*, \lambda^*)}{\partial f_k^{rs}} \geq 0 \quad \dfrac{\partial L(f^*, \lambda^*)}{\partial \lambda_{rs}} = 0$

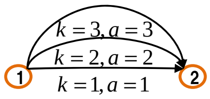
Lagrangian function

$$L(f, \lambda) = Z_p(f) - \sum_{rs} \lambda_{rs} \{ \sum_k f_k^{rs} - Q_{rs} \} \quad f_k^{rs} \geq 0$$

User Equilibrium with Fixed Demand

Example

Problem



Cost function

$t_1 = 5 + 0.1 f_1$

$t_2 = 10 + 0.025 f_2$

$t_3 = 15 + 0.025 f_3$

$Q_{12} = 200$

f [pcu/min]

pcu: passenger car unit

t [min]

Optimal problem

$$L(f, \lambda) = 5 f_1 + 0.05 f_1^2 + 10 f_2 + 0.0125 f_2^2 + 15 f_3 + 0.0125 f_3^2$$

$$\text{s.t.} \ \sum_{k=1}^{3} f_k = 200 \quad f_k \geq 0 (k = 1 \sim 3)$$

Karush-Kuhn-Tucker condition

$$L(f, \lambda) = 5 f_1 + 0.05 f_1^2 + 10 f_2 + 0.0125 f_2^2 + 15 f_3 + 0.0125 f_3^2 - \lambda(\sum_{k=1}^{3} f_k - 200) \quad f_k \geq 0 (k = 1 \sim 3)$$