Optimization Approaches to Power Networks

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Outline

A Simple Optimal Power Flow Model with Energy Storage
K. Mani Chandy, Steven H Low, Ufuk Topcu and Huan Xu
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- Background
- Model and Problem Formulation
- Single Generator Single Load
- Network Case

Background

The optimal power flow (OPF) problem

- to optimize a objective over power network under constraints
  Minimization of generation cost
- Classical OPF : without storage

Renewable energy, such as wind power, is intermittent.

⇨ Integration of renewable energy into electric grid is difficult.
Battery is necessary!

Goal
To understand the impact of storage on optimal generation schedule

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OPF model

admittance matrix : Y potential : V phase : θ

The (real) power flow $i \to j$:

$$V_iV_jY_{ij}\sin(\theta_i(t)-\theta_j(t))$$

$$\approx V_iV_jY_{ij}(\theta_i(t)-\theta_j(t)) \le \bar{q}_{ij}(t) \quad \cdots(1)$$

Line capacities

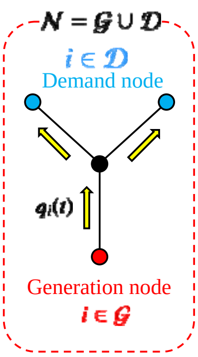
$\because |\theta_i(t)-\theta_j(t)|$ is small

Net power export (from Kirchhoff's laws):

$$q_i(t)=\sum_{j\in N} V_iV_jY_{ij}(\theta_i(t)-\theta_j(t)) \quad \cdots(2)$$

$q_i(t)>0$:supply $q_i(t)<0$:consume

Demand node: $q_i(t)=-d_i(t) \quad \cdots(3)$



OPF model

Each generation node :

$$q_i(t)=g_i(t)+r_i(t) \quad \cdots(4)$$

Generator produce Battery charge or discharge

$$g_i(t)\ge 0 \quad \cdots(5)$$

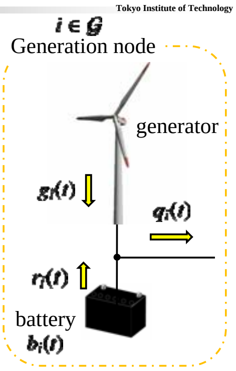
$r_i(t)<0$: charge $r_i(t)>0$: discharge

The battery energy level:

$$b_i(t)=b_i(t-1)-r_i(t) \quad \cdots(6)$$

$$0\le b_i(t)\le B_i \quad \cdots(7)$$

Generation cost : $c_i(g_i,t)$
Battery cost : $h_i(b_i,r_i)$



⇨ Battery charge when the cost of generation is high, and discharge when it is low.

OPF problem

OPF problem with energy storage

$$\min_{\theta,q,r,g,b} \sum_{t=1}^{T} \sum_{i \in G} [c_i(g_i(t), t) + h_i(b_i(t), r_i(t))]$$

Generation cost — Battery cost

s.t. (1), (2), (3), (4), (5), (6), (7)

for $t = 1, \ldots, T$

$c_i(g_i(t), t) := \frac{1}{2}\gamma_i(t) g_i^2(t)$: convex function $\cdots (*)$

$\gamma_i(t)$: the time-varying nature of $c_i(g_i(t), t)$

$h_i(b_i, r_i) = h_i(b_i)$: strictly decreasing

Ex. $h_i(b_i) = \alpha_i (B_i - b_i)$ $\alpha_i > 0$, $B_i > 0$

⇒ OPF-S problem is convex program

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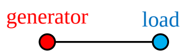
SGSL model

(2), (3), (4) ⇒ $g^*(t) + r^*(t) = d(t)$

OPF-S

$$\min \sum_{t=1}^{T} c(g(t), t) + h(b(t))$$



s.t. $b(t) = b(t-1) - d(t) + g(t)$ [dual var : $\tilde{b}(t)$]

$g(t) \geq 0$ [dual var : $\hat{\lambda}(t)$]

$b(t) \geq 0$ [dual var : $\underline{b}(t)$]

$B - b(t) \geq 0$ [dual var : $\bar{b}(t)$]

for $t = 1, \ldots, T$

⇒ Convex program

⇒ KKT condition is both necessary and sufficient for optimality

KKT condition

- $\frac{\partial h}{\partial b(t)} + \tilde{b}(t) - \tilde{b}(t+1)\mathbf{1}(t < T) = \underline{b}(t) - \bar{b}(t)$

  ⇒ $\tilde{b}(t) = H^*(t) + B^*(t)$ $\mathbf{1}$: indicator function

  $H^*(t) := \sum_{\tau=t}^{T} -\frac{dh_t}{db(\tau)}(b^*(\tau)) > 0$, $B^*(t) := \sum_{\tau=t}^{T} (\underline{b}(\tau) - \bar{b}(\tau))$

  → the marginal storage cost-to-go

- $\gamma(t) g^*(t) = \tilde{b}(t) + \hat{\lambda}(t)$, $\hat{\lambda}(t) g^*(t) = 0$

→ Optimal solution

$$g^*(t) = \left[\bar{g}(t) + \frac{B^*(t)}{\gamma(t)}\right]^+$$

$$b^*(t) = b^*(t-1) - d(t) + \left[\bar{g}(t) + \frac{B^*(t)}{\gamma(t)}\right]^+$$

Nominal generation : $\bar{g}(t) := \frac{H^*(t)}{\gamma(t)}$ $[x]^+ := \max(x, 0)$

Optimal solution (SGSL)

➢The case where battery constraint is inactive

$b(t) \in (0, B)$ ⇒ $B^*(t) \equiv 0$, $g^* = \bar{g}$

$$\gamma(t) g^*(t) = H^*(t) = \sum_{\tau=1}^{T} -\frac{dh}{db(\tau)}(b^*(\tau))$$

Marginal generation cost — Marginal storage cost-to-go

⇒ $\gamma(t) g^*$ strictly decreases $\because \frac{dh}{db} < 0$

Assumptions

A0 : $d(t) > 0, \gamma(t) > 0$, for $t = 1, \ldots, T$. For all $b \geq 0$, $\frac{dh(b)}{db} < 0$

A1 : $\gamma(t) d(t) - \gamma(t+1) d(t+1) < -\frac{dh(b(t))}{db(t)}$, for $t = 1, \ldots, T$

$\gamma(t) g^*$ will decrease at the same rate at which $H^*(t)$ decreases

Optimal solution (SGSL)

➢The case which battery constraint is active : $b(t) \in [0, B]$

Under A1 ⇒ Optimal generation schedule has 3 phase

1 : $g^*(t) > d(t)$ and the battery charges

2 : $g^*(t) = d(t)$ and the battery remains saturated

3 : $g^*(t) < d(t)$ and the battery discharges

Definition

$\bar{d} := \frac{1}{T}\sum_{t=1}^{T} d(t)$ $\bar{\gamma} := \left(\frac{1}{T}\sum_{t=1}^{T} \frac{1}{\gamma(t)}\right)^{-1}$

$\bar{d}_1 := \frac{1}{l}\sum_{t=1}^{l} d(t)$ $\bar{\gamma}_1 := \left(\frac{1}{l}\sum_{t=1}^{l} \frac{1}{\gamma(t)}\right)^{-1}$

$\bar{d}_3 := \frac{1}{T-m}\sum_{t=m+1}^{T} d(t)$ $\bar{\gamma}_3 := \left(\frac{1}{T-m}\sum_{t=m+1}^{T} \frac{1}{\gamma(t)}\right)^{-1}$

Theorem 1

Suppose $0 < b(0) < B$. The optimal generation g^* crosses the demand curve $d(t)$ at most once, from above, the optimal battery level b^* is unimodal.

1) battery never saturates

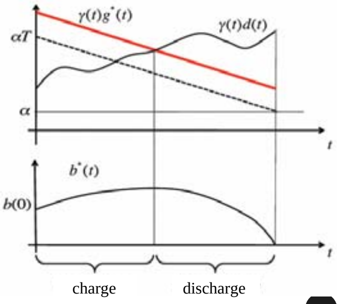
$$g^*(t) = \bar{g}(t) + \frac{\bar{\gamma}}{\gamma(t)}[\bar{d} - \sigma]^+$$

$$[x]^+ := \max(x, 0)$$

$$\sigma := \frac{1}{T}\sum_{t=1}^{T}\bar{g}(t) + \frac{b(0)}{T}$$

$$b^*(T) = T[\sigma - \bar{d}]^+$$

Total available energy : $T\sigma$



Theorem 1

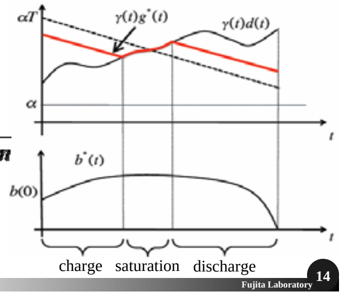
2) battery saturates

$$g^*(t) = \begin{cases} \bar{g}(t) - \frac{\bar{\gamma}_1}{\gamma(t)}(\sigma_1 - \bar{d}_1), & t = 1, \ldots, l \\ d(t), & t = l+1, \ldots, m \\ \bar{g}(t) + \frac{\bar{\gamma}_3}{\gamma(t)}[\bar{d}_3 - \sigma_3]^+, & t = m+1, \ldots, T \end{cases}$$

$$\sigma_1 := \frac{1}{l}\sum_{t=1}^{l}\bar{g}(t) - \frac{B - b(0)}{l}$$

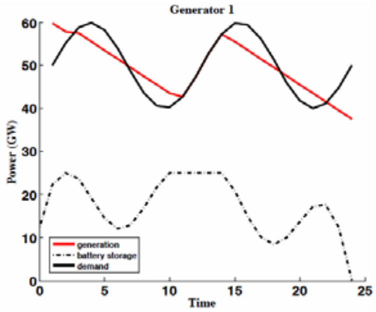
$$\sigma_3 := \frac{1}{T-m}\sum_{t=m+1}^{T}\bar{g}(t) - \frac{B}{T-m}$$

$$b^*(T) = (T-m)[\sigma_3 - \bar{d}_3]^+$$



SGSL example (violating A1)

- $T = 24$ hours
- $d(t) = 10 \cdot \sin\frac{4\pi}{T-1}(t-1)$
- $B = 25$ GW
- $b(0) = 12.5$ GW
- $\gamma(t) \equiv 1$ ⇒ $c = \frac{1}{2}g^2$
- $h(b) = 2(B - b)$



Demand decreases faster than that required for A1 to hold

⇒ b^* discharges and recharge twice more before reaching 0 at T

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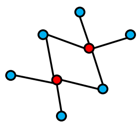
Network case (MGML)

Multiple generation and multiple loads ⇒ the problem in (*)

$$g_i(t) + r_i(t) = \sum_{j \in N} Y_{ij}(\theta_i(t) - \theta_j(t)), \ i \in G$$

$$-d_i(t) = \sum_{j \in N} Y_{ij}(\theta_i(t) - \theta_j(t)), \quad i \in D$$



Optimal generation : $g^*(t) = (\mathrm{diag}(\gamma_i(t)))^{-1}[H^*(t) + B^*(t)]^+$

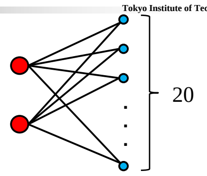
$$H^*(t) = \mathrm{diag}\left(\sum_{\tau=t}^{T} -\frac{dh_i}{db(\tau)}(b_i^*(\tau))\right) \quad B^*(t) = \mathrm{diag}\left(\sum_{\tau=t}^{T}(\underline{b}_i(\tau) - \bar{b}_i(\tau))\right)$$

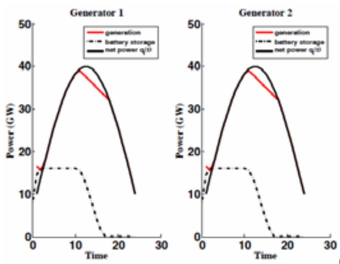
Effect of network : $\begin{bmatrix} g^*(t) + r^*(t) \\ -d(t) \end{bmatrix} = \mathbf{Y}\theta^*(t)$, $\mathbf{Y}_{ij} = \begin{cases} \sum_{j \in N} Y_{ij}, & i = j \\ -Y_{ij}, & i \neq j \end{cases}$

Network case example1

➢ Symmetric network with 2 generator

- $Y_{ij} = 1$
- $\bar{q}_{ij} = 1000$ GW
- $T = 24$ hours
- $B_i = 16$ GW
- $b_i(0) = 8$ GW
- $d_i(t) = 3\sin\frac{\pi t}{T} + 1$
- $h_i(b_i) = B_i - b_i$
- $\gamma_i(t) = 1$





Network case example2

➢Cost savings (SGSL case and network case)

➡ How much cost can be saved compared to total cost without battery?

- $\gamma_i(t) \equiv \gamma$
- time varying $\gamma_i(t)$

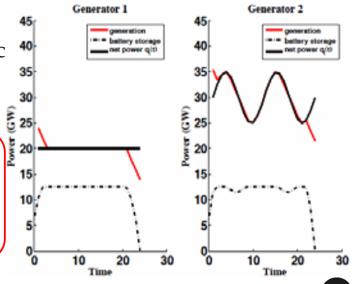
	saving for SGSL	saving for MGML
time-invariant γ_i	98%	97%
time-varying γ_i	83%	85%

Link capacities are not symmetric

➡ Generator 1 < Generator 2

➢With time-varying γ_i, the cost savings are higher

➢Battery is more valuable in the presence of fluctuations



Relationship Between Power Loss and Network Topology

Relationship Between Power Loss and Network Topology in Power systems

Javad Lavaei and Steven H Low

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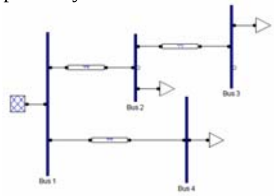
CDC

➢How the minimum power loss in a power system is related to its network topology?

$$\min\ \mathrm{Re}\{V_n I_n^*\} - \sum_{k=1}^{n-1} P_k$$
$$\text{s.t. } V_k I_k^* = -P_k - Q_k i$$
$$|V_n| = V_0$$
$$I = YV$$



➡ P_{loss} : the minimum of the active power loss

LMI Optimal Problem

LMI optimization problem

$$\max\ f(\lambda,\bar{\lambda},\mu) := \sum_{k=1}^{n-1} \lambda_k P_k + \sum_{k=1}^{n-1} \bar{\lambda}_k Q_k - \mu V_0^2$$
$$\text{s.t. } \Phi(\lambda,\bar{\lambda},\mu) := \sum_{k=1}^{n-1} \lambda_k Y_k + \sum_{k=1}^{n-1} \bar{\lambda}_k \bar{Y}_k + Y + \mu M \geq 0$$

➡ $P_{\min}$

weak duality theorem ➡ $0 \leq P_{\min} \leq P_{\mathrm{loss}}$

(The same applies to Q)

➢The feasibility region is defined only by the network topology

➡ makes it possible to address many important power problems

Outline

A Majorization-Minimization Approach to Design of Power Transmission Networks

Jason K. Johnson and Michael Cherkov

Los Alamos National Laboratory

April 13, 2010

- Background
- Network Optimization Problem
- Selecting Network Structure
- Robust Network Design

Background

The power grid of today was not systematically planned but grew in a piecemeal fashion.

➡ The greatest engineering achievements of the 20th century

However…

➢This status quo is now challenged with increased demand and stress on the aging network.

➢A shift towards renewable source energy will further stress the grid.

➡ It is important to incorporate new and extend existing infrastructure in a systematic way.

Suggest an efficient algorithmic approach for optimal or close to optimal power grid design

The Network Optimization Problem

Goal : to assign conductance θ to balance

- (1) maximizing network efficiency
- (2) minimizing the cost of building the network

The cost of building the network :

$$\alpha^T\theta = \sum_l \alpha_l \theta_l$$
$$\alpha_l = c g^{-1} s_l^2$$

c : price of copper

g : conductivity of copper

s_l : total length of line l

Optimization problem

$$\min\ \mathcal{L}(\theta)$$
$$\text{s.t. } \theta \geq 0$$
$$\alpha^T\theta \leq C$$

power loss (convex)

➡ $\min_{\theta \geq 0} \{\mathcal{L}(\theta) + \lambda \alpha^T \theta\}$

penalty

C : budget

$\lambda > 0$: Lagrange multiplier

Convex Optimization Algorithm

graph $\boldsymbol{G}$: $\boldsymbol{w} \times \boldsymbol{w}$ grid of nodes

lines between nearest and 2nd nearest neighbors

$$\min_{\theta > 0}\{\mathcal{L}(\theta) + \alpha^T\theta - \zeta \sum_{l \in G} \log \theta_l\} \xrightarrow{\zeta \to 0} \min_{\theta \geq 0}\{\mathcal{L}(\theta) + \alpha^T\theta\} \quad \lambda = 1$$

Log-barrier function

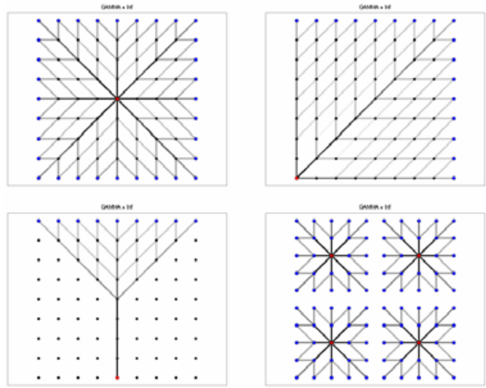
➡ The solution will always be strictly positive.

Algorithm

updating iteratively the solution :

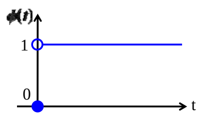
➡ large ζ ⟶ small ζ

Demonstrations



Sparsity-Favoring Network Cost

$$\min_{\theta \geq 0}\{\mathcal{L}(\theta) + \alpha^T\theta + \beta^T\phi(\theta)\}$$



$\phi(t)$: unit-step function, concave

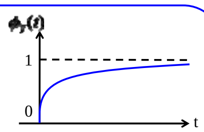
discontinuous

$\beta^T\phi(\theta)$: other cost (ex. labor)

Annealed Smoothing Method

$$\phi_\gamma(t) = \frac{t}{t+\gamma}, \quad \gamma > 0$$

: concave, continuous



"deterministic annealing" strategy

large γ : $\phi_\gamma(t) \approx 0$ ➡ convex optimization problem

↓ update iteratively

small γ : $\phi_\gamma(t) \to \phi(t)$ ➡ difficult combinatorial problem

Majorization-Minimization Algorithm

objective function : $f(x) = f_\cup(x) + f_\cap(x)$

convex concave

linear upper-bound of $f_\cap(x)$:

$$f_\cap(x) \leq f_\cap(x^{(k)}) + \nabla f_\cap(x^{(k)})^T(x - x^{(k)})$$

previous guess of the solution

➡ $f(x) \leq f_\cup(x) + \nabla f_\cap(x^{(k)})^T x + \text{const}$ ← independent of x

convex function

➡ $x^{(k+1)} = \arg\min_x\{f_\cup(x) + \nabla f_\cap(x^{(k)})^T x\}$

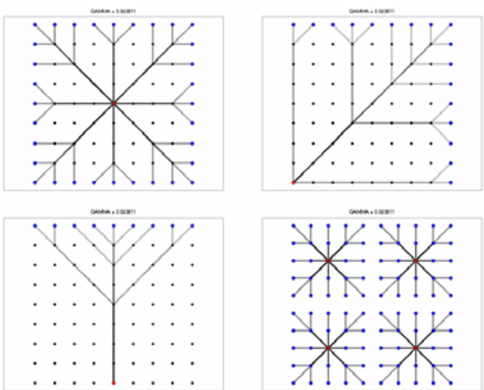
$$\begin{cases} x = \theta \\ f_\cup(\theta) = \mathcal{L}(\theta) + \alpha^T\theta \\ f_\cap(\theta) = \beta^T\phi_\gamma(\theta) \end{cases}$$

➡ **iterative algorithm**

$$\alpha_l^{(k)} = \alpha_l + \frac{\gamma}{(\gamma + \theta_l^{(k-1)})^2}\beta_l$$

$$\theta^{(k)} = \arg\min_{\theta \geq 0}\{\mathcal{L}(\theta) + (\alpha^{(k)})^T\theta\}$$

Demonstrations (Sparsity-Favor)



Robust Network Design

$z \in \{0, 1\}^m$: indicator vector of line failing

operational failed

power dissipation after removing failed lines : (all entries one)

$$\mathcal{L}(\theta; z) \triangleq \mathcal{L}((\mathbf{1} - z) \circ \theta)$$

The worst case power dissipation : $\mathcal{L}^{\backslash k}(\theta) = \max_{\mathbf{1}^T z = k} \mathcal{L}(\theta; z)$ ← non-smooth

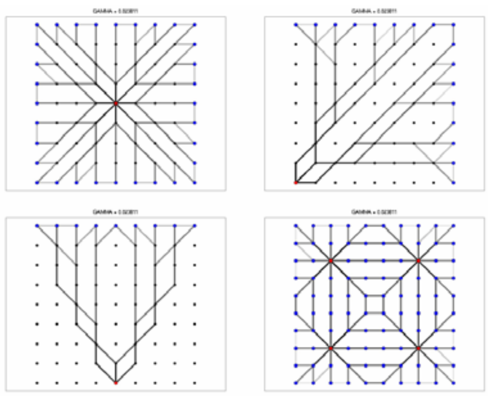
➡ $\min_{\theta \geq 0}\{\mathcal{L}^{\backslash k}(\theta) + \alpha^T\theta + \beta^T\phi(\theta)\}$

Gibbsian "Soft-Max" Optimization

$$\mathcal{L}_\tau^{\backslash k}(\theta) = \tau \log \sum_{\mathbf{1}^T z = k} \exp[\tau^{-1}\mathcal{L}(\theta; z)], \quad \tau > 0$$

➡ smooth, convex, upper-bound to $\mathcal{L}^{\backslash k}(\theta)$, $\mathcal{L}_\tau^{\backslash k} \to \mathcal{L}^{\backslash k}$ $(\tau \to 0)$

Demonstrations (Robust Network)



Appendix

Karush-Kuhn-Tucker (KKT) conditions

$$\max_x f(x) \quad \text{s.t. } g_i(x) \le 0, \quad h_j(x) = 0$$

the Lagrangian

$$L(x,\lambda,\mu) = f(x) - \sum_i \lambda_i g_i(x) + \sum_j \mu_j h_j(x)$$

Suppose there exists constants $\lambda_i \ge 0$ and μ_j such that

$$\frac{\partial f}{\partial x_k}(x^*) - \sum_i \lambda_i \frac{\partial g_i}{\partial x_k}(x^*) + \sum_j \mu_j \frac{\partial h_j}{\partial x_k}(x^*) = 0 \quad , \forall k$$

$$\lambda_i g_i(x^*) = 0 \quad , \forall i$$

⇒ the first-order necessary condition for optimality

Resistive Network Model

G : graph with node set $N = \{1, ..., n\}$ and m (undirected) edges $\{i, j\} \in G \subset 2^N$

edge weights $\theta_{ij} \equiv \theta_{ji} \ge 0$ ⟶ conductance

Conductance matrix : $K_{ij}(\theta) = \begin{cases} -\theta_{ij}, & i \ne j \\ \sum_{k \ne i} \theta_{ik}, & i = j \end{cases}$ standard basis vector

⇒ $K(\theta) = A\mathrm{Diag}(\theta)A^T$, $\mathrm{Diag}(\theta) = \sum_i \theta_i e_i e_i^T$

$b \in \mathbb{R}^N$: the vector of injected currents

$b_i > 0$: sources , $b_i = 0$: transmission , $b_i < 0$: sinks

$u \in \mathbb{R}^N$: electric potential

⇒ $Ku = b$

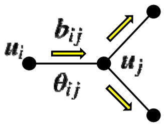
- $u^T K u \ge 0$
- has a single zero eigenvalue : $K\mathbf{1} = 0$ (all entries one)
- $Ku = \lambda u$ ⇒ $\mathbf{1}^T u = 0$, $\lambda > 0$

Resistive Network Model

To determine the unique solution

$$u = K^+ b \,, \quad K^+ \triangleq (K + \mathbf{1}\mathbf{1}^T)^{-1}$$

the current flow from i to j : $b_{ij} = \theta_{ij}(u_i - u_j)$



The total power loss over the network :

$$\mathcal{L} = \sum_{ij \in G} \theta_{ij}(u_i - u_j)^2 = u^T K u \xrightarrow{u = K^+ b} b^T K^+ b$$

If G and b are fixed : $\mathcal{L}(\theta) = b^T (K(\theta) + \mathbf{1}\mathbf{1}^T)^{-1} b$

The expected power loss for a random current :

$$\mathcal{L}(\theta) = \langle b^T K^+(\theta) b \rangle = \langle \mathrm{Tr}(K^+(\theta) b b^T) \rangle$$

$$= \mathrm{Tr}(K^+(\theta)\langle b b^T \rangle) = \mathrm{Tr}(K^+(\theta) B) \qquad B \triangleq \langle b b^T \rangle$$

⇒ convex function

DC Approximation to AC Power Flow

The (DC) resistive network can be used to approximate the AC system.

u_j ⇒ complex potential : $U_j = \exp(i\varphi_j)$

real φ_j : the phase of the potential

➢ Susceptance is normally an order of magnitude larger than conductance

DC-approximation of AC Kirchhoff equations

$p = \tilde{K}\varphi$ — p: the vector of real power; $\tilde{K}$: network susceptance matrix

first order in the conductance-to-susceptance ratio

⇒ $\mathcal{L} = \frac{1}{2} p^T \tilde{K}^+ K \tilde{K}^+ p$, $\tilde{K}^+ \triangleq (\tilde{K} + \mathbf{1}\mathbf{1}^T)^{-1}$

conductance-to-admittance ratio μ

⇒ $\tilde{K} = \frac{1}{\mu} K$ ⇒ $\mathcal{L}(\theta) = \frac{\mu^2}{2} \mathrm{Tr}(\tilde{K}^+ B)$