Convergence Analysis of Visual Attitude Synchronization

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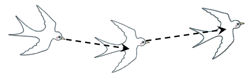
Objective

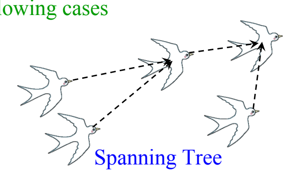
Visual Attitude Synchronization

Synchronize all agents' attitudes using only visual information
(not using centralized sensing devices or communication)

In this presentation, consider the following cases



Chain



Spanning Tree

Bidirectional

Outline

- Introduction
- Problem Setting
- Convergence Analysis
  - Chain Type Visibility (Information Structure)
  - Spanning Tree Type Visibility
- Difficulty of Convergence
  - Mutual Visibility
- Future Works

Rigid Body Motion in *SE*(3)

Position and Orientation

$g_{wi} = \begin{bmatrix} e^{\hat{\xi}\theta_{wi}} & p_{wi} \\ 0 & 1 \end{bmatrix} \in \mathcal{R}^{4\times 4}$

$p_{wi} \in \mathcal{R}^3$ $\quad e^{\hat{\xi}\theta_{wi}} \in SO(3)$ $\quad (p_{wi}, e^{\hat{\xi}\theta_{wi}}) \in SE(3)$

Exponential Coordinate for Rotation
- $\xi_{wi} \in \mathcal{R}^3$: Rotation Axis ($\|\xi_{wi}\| = 1$)
- $\theta_{wi} \in \mathcal{R}$: Rotation Angle ($|\theta_{wi}| < \pi$)

"∧" (wedge) : $\mathcal{R}^3 \to so(3)$

$$\hat{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

"∨" (vee) : $so(3) \to \mathcal{R}^3$ (Inverse Operator)

Body Velocity

$$V_{wi}^b = \begin{bmatrix} v_{wi}^b \\ \omega_{wi}^b \end{bmatrix} \in \mathcal{R}^6 \quad \overset{\text{wedge}}{\underset{\text{vee}}{\rightleftarrows}} \quad \hat{V}_{wi}^b = g_{wi}^{-1}\dot{g}_{wi} = \begin{bmatrix} \hat{\omega}_{wi}^b & v_{wi}^b \\ 0 & 0 \end{bmatrix} \in \mathcal{R}^{4\times 4}$$

- $V_{wi}^b \in \mathcal{R}^6$: Body Velocity
- $v_{wi}^b \in \mathcal{R}^3$: Linear Velocity
- $\omega_{wi}^b \in \mathcal{R}^3$: Angular Velocity

Rigid Body Motion in *SE*(3)

$$\dot{g}_{wi} = g_{wi}\hat{V}_{wi}^b \quad (1)$$

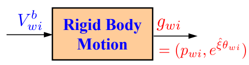


Fig. 2: Block Diagram of Rigid Body Motion

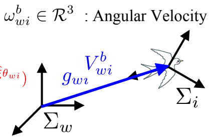


Fig. 1: Coordinate of Rigid Body in *SE*(3)

[1] R. Murray, Z. Li and S. S. Sastry, *Mathematical Introduction to Robotic Manipulation*, CRC Press, 1994.
[2] Y. Ma, S. Soatto, J. Koseck and S. S. Sastry, *An Invitation to 3-D Vision: From Images to Geometric Models*, Springer, 1st ed. 2004.

Passivity of Rigid Body Motion

Rigid Body Motion

$$\dot{g}_{wi} = g_{wi}\hat{V}_{wi}^b$$

Vector Representation of g_{wi}

$$e_{wi} := \begin{bmatrix} p_{wi} \\ \mathrm{sk}(e^{\hat{\xi}\theta_{wi}})^\vee \end{bmatrix}$$

Transformation from Σ_w to Σ_i

$$\Pi_{wi} = \begin{bmatrix} e^{-\hat{\xi}\theta_{wi}} & 0 \\ 0 & e^{-\hat{\xi}\theta_{wi}} \end{bmatrix} e_{wi}$$

Lemma 1 (Passivity)
The rigid body motion (1) satisfies

$$\int_0^T (V_{wi}^b)^T \Pi_{wi} dt \geq -\beta_i, \ {}^\forall T > 0$$

where β_i is a positive scalar.

Error Function of Rotation Matrix

$$\phi(e^{\hat{\xi}\theta_{wi}}) = \frac{1}{2}\mathrm{tr}(I_3 - e^{\hat{\xi}\theta_{wi}}) \quad (= 1 - \cos\theta_{wi})$$

Propety:
- $\phi(e^{\hat{\xi}\theta_{wi}}) \geq 0$
- $\phi(e^{\hat{\xi}\theta_{wi}}) = 0 \Leftrightarrow e^{\hat{\xi}\theta_{wi}} = I_3$
- $\dot{\phi}(e^{\hat{\xi}\theta_{wi}}) = (\mathrm{sk}(e^{\hat{\xi}\theta_{wi}})^\vee)^T \omega_{wi}^b$

(Proof) Differentiate the following energy function w.r.t. time.

$$E(g_{wi}) := \frac{1}{2}\|p_{wi}\|^2 + \phi(e^{\hat{\xi}\theta_{wi}})$$

Passivity: $V_{wi}^b \Rightarrow \Pi_{wi}$

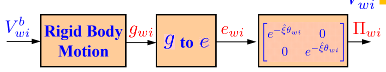


Fig. 3: Block Diagram of Passivity of Rigid Body Motion

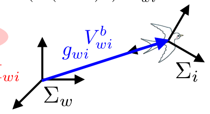


Fig. 1: Coordinate of Rigid Body in *SE*(3)

Networked Passive Systems

n Passive Systems

$$\dot{g}_{wi} = g_{wi}\hat{V}_{wi}^b, \quad i \in \{1, \cdots, n\}$$

Restriction of Information Structure

Information Flow Model

$G := (\mathcal{V}, \mathcal{E})$: Graph

$\mathcal{V} := \{1, \cdots, n\}$: Set of rigid bodies

$\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$: Set of information flow

Neighborhood $\mathcal{N}_i := \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$
:Set of rigid bodies whose information is available to rigid body i

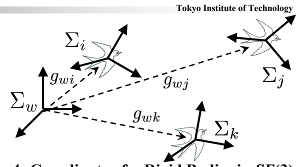


Fig. 4: Coordinate of *n* Rigid Bodies in *SE*(3)

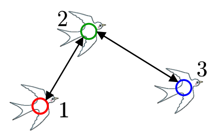


Fig. 5: Graph Topology

Visual Attitude Synchronization

Visual Attitude Synchronization

$$\begin{cases} \lim_{t\to\infty} \phi(e^{\hat{\xi}\bar{\theta}_{ij}}) = 0 \\ \lim_{t\to\infty} E(g_{eeij}) = 0 \end{cases} \quad {}^{\forall} j \in \mathcal{N}_i,\ {}^{\forall} i \in \{1,\cdots,n\} \quad (2)$$

$$\left(E(g_{ij}) = \frac{1}{2}\|p_{ij}\|^2 + \phi(e^{\hat{\xi}\theta_{ij}}) = 0 \ \Rightarrow\ g_{wi} = g_{wj} \right)$$

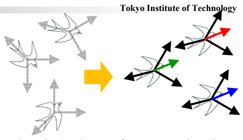


Fig. 6: Attitude Synchronization

$$\begin{cases} \bar{g}_{ij} = (\bar{p}_{ij}, e^{\hat{\xi}\bar{\theta}_{ij}}) \ \text{: Estimated Relative Pose} \\ g_{eeij} := \bar{g}_{ij}^{-1} g_{ij} \ (= (p_{eeij}, e^{\hat{\xi}\theta_{eeij}})) \ \text{: Estimation Error} \end{cases}$$

➡ Estimate Relative Pose by Nonlinear Observer (Visual Motion Observer)

- Estimated Relative Attitude: I_3
- Estimation Error: 0 $(\bar{g}_{ij} = g_{ij})$

} Attitude Synchronization

System Representation

Control Error: $e^{\hat{\xi}\theta_{ecij}} := e^{\hat{\xi}\bar{\theta}_{ij}}$

Estimation Error: $g_{eeij} := \bar{g}_{ij}^{-1} g_{ij}$

Vector ➡

$$e_{cij} = \mathrm{sk}(e^{\hat{\xi}\theta_{ecij}})^{\vee} \in \mathcal{R}^3$$

$$e_{eij} = \begin{bmatrix} p_{eeij} \\ \mathrm{sk}(e^{\hat{\xi}\theta_{eeij}})^{\vee} \end{bmatrix} \in \mathcal{R}^6$$

System Dynamics

Control Error System (Only Attitude)

$$\omega_{ecij}^b = -e^{-\hat{\xi}\theta_{ecij}} \omega_{wi}^b + u_{weij} \quad (3)$$

Estimation Error System (Observer)

$$V_{eeij}^b = -\mathrm{Ad}_{(g_{eeij}^{-1})} u_{eij} + V_{wj}^b \quad (4)$$

$$\omega_{ecij}^b = (e^{\hat{\xi}\theta_{ecij}} \dot{e}^{\hat{\xi}\theta_{ecij}})^{\vee} \in \mathcal{R}^3$$

$$V_{eeij}^b = (g_{eeij}^{-1} \dot{g}_{eeij})^{\vee} \in \mathcal{R}^6$$

$$u_{eij} = \begin{bmatrix} u_{veij} \\ u_{\omega eij} \end{bmatrix} \in \mathcal{R}^6$$

: i' s Control Input of Observer

$\mathrm{Ad}_{(g_{ij}^{-1})}$: Coordinate Transformation

$$\left(\mathrm{Ad}_{(g_{ij})} = \begin{bmatrix} e^{\hat{\xi}\theta_{ij}} & \hat{p}_{ij} e^{\hat{\xi}\theta_{ij}} \\ 0 & e^{\hat{\xi}\theta_{ij}} \end{bmatrix} \in \mathcal{R}^{6\times 6} \right)$$

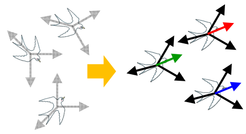


Fig. 6: Attitude Synchronization

Control Input

Control Input

Angular Velocity Input

$$\omega_{wi}^b = \sum_{j\in\mathcal{N}_i} K_{ij} \mathrm{sk}(e^{\hat{\xi}\bar{\theta}_{ij}})^{\vee}, \ i \in \{1,\cdots,n\} \quad (5) \qquad K_{ij} > 0$$

Estimated Relative Attitude

Control Input for Visual Motion Observe

$$u_{eij} = K_{eij} \left(e_{eij} - \begin{bmatrix} 0_{3\times 1} \\ \mathrm{sk}(e^{\hat{\xi}\bar{\theta}_{ij}})^{\vee} \end{bmatrix} \right), \ j \in \mathcal{N}_i, \ i \in \{1,\cdots,n\} \quad (6)$$

Influence of Velocity Input $\qquad K_{eij} > 0$

e_{eij} is available from image feature points (Appendix)

Assumptions

(A1) $v_{wi}^b = 0 \ {}^{\forall} i \in \{1,\cdots,n\}$

(A2) $\theta_{eeij} \ {}^{\forall} j \in \mathcal{N}_i, \ {}^{\forall} i \in \{1,\cdots,n\}$ are small enough (for Observer)

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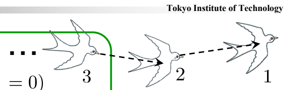
Convergence Analysis (Chain Type)

Assumptions

(A3) Chain Type Visibility

(A4) Lead rigid body doesn't move$(\omega_{w1}^b = 0)$

(A5) $K_{(i-1)(i-2)} < \dfrac{2K_{i(i-1)}K_{ei(i-1)}}{K_{i(i-1)} + 2K_{ei(i-1)}}$, $i \in \{3,\cdots,n\}$



Chain Type

Theorem 1 (Visual Attitude Synchronization)

Consider the n rigid bodies with visual motion observer represented by (1) and (4). Then, under the assumptions A1-A5, the control input (5), (6) achieves attitude synchronization in the sense of (2).

(Proof) 3 Rigid Bodies (for explanation)

System Dynamics

$$\begin{bmatrix} \omega_{ec21}^b \\ \omega_{ec32}^b \\ V_{ee21}^b \\ V_{ee32}^b \end{bmatrix} = \begin{bmatrix} -e^{\hat{\xi}\theta_{ec21}} & 0 & 0 & I_3 & 0 & 0 \\ 0 & -e^{\hat{\xi}\theta_{ec32}} & 0 & 0 & 0 & I_3 \\ 0 & 0 & -\mathrm{Ad}_{(g_{ee21}^{-1})} & & 0 & \\ 0 & 0 & 0 & & -\mathrm{Ad}_{(g_{ee32}^{-1})} & \end{bmatrix} \begin{bmatrix} \omega_{w2}^b \\ \omega_{w3}^b \\ u_{e21} \\ u_{e32} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ V_{w1}^b \\ V_{w2}^b \end{bmatrix}$$

Convergence Analysis (Chain Type)

Energy Function:

$$E_3 := \phi(e^{\hat{\xi}\theta_{ec21}}) + \phi(e^{\hat{\xi}\theta_{ec32}}) + E(g_{ee21}) + E(g_{ee32}) \quad (7)$$

Differentiate (7) w.r.t. time

$$\begin{aligned} \dot{E}_3 &= (\mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^{\vee})^T \omega_{ec21}^b + (\mathrm{sk}(e^{\hat{\xi}\theta_{ec32}})^{\vee})^T \omega_{ec32}^b \ (3) \\ &\quad + e_{e21}^T \mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee21}})} V_{ee21}^b + e_{e32}^T \mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee32}})} V_{ee32}^b \ (4) \\ &= -e_{c21}^T \omega_{w2}^b - e_{c32}^T \omega_{w3}^b - e_{e21}^T u_{e21} - e_{e32}^T u_{e32} + e_{c21}^T u_{\omega e21} \\ &\quad + e_{c32}^T u_{\omega e32} + e_{e21}^T \mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee21}})} V_{w1}^b + e_{e32}^T \mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee32}})} V_{w2}^b \\ &= -\underbrace{[e_{c21}^T \ e_{c32}^T \ e_{e21}^T \ e_{e32}^T]}_{=:\, e_3^T} \underbrace{\begin{bmatrix} I_3 & 0 & 0 \ -I_3 & 0 \\ 0 & I_3 & 0 & 0 \ -I_3 \\ 0 & 0 & I_6 & 0 \\ 0 & 0 & 0 & I_6 \end{bmatrix}}_{=:\, N_3^T} \underbrace{\begin{bmatrix} \omega_{w2}^b \\ \omega_{w3}^b \\ u_{e21} \\ u_{e32} \end{bmatrix}}_{=:\, u_3} \\ &\quad + e_{e21}^T \mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee21}})} V_{w1}^b + e_{e32}^T \mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee32}})} V_{w2}^b \end{aligned}$$

Convergence Analysis (Chain Type)

$$\dot{E}_3 = -e_3^T N_3^T u_3 + e_{e21}^T \mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee21}})} V_{w1}^b + e_{e32}^T \mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee32}})} V_{w2}^b$$

Control Input

$$u_3 = K_3 N_3 e_3 = \begin{bmatrix} K_{21}\mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^\vee \\ K_{32}\mathrm{sk}(e^{\hat{\xi}\theta_{ec32}})^\vee \\ K_{e21}\left(\begin{bmatrix} p_{ee21} \\ \mathrm{sk}(e^{\hat{\xi}\theta_{ee21}})^\vee \end{bmatrix} - \begin{bmatrix} 0 \\ \mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^\vee \end{bmatrix}\right) \\ K_{e32}\left(\begin{bmatrix} p_{ee32} \\ \mathrm{sk}(e^{\hat{\xi}\theta_{ee32}})^\vee \end{bmatrix} - \begin{bmatrix} 0 \\ \mathrm{sk}(e^{\hat{\xi}\theta_{ec32}})^\vee \end{bmatrix}\right) \end{bmatrix} \quad K_3 = \begin{bmatrix} K_{21}I_3 & 0 & 0 & 0 \\ 0 & K_{32}I_3 & 0 & 0 \\ 0 & 0 & K_{e21}I_6 & 0 \\ 0 & 0 & 0 & K_{e32}I_6 \end{bmatrix}$$

$=: e_{ee32}$

$$\left(\mathrm{sk}(e^{\hat{\xi}\theta})^\vee = \xi \sin\theta \Rightarrow \|\mathrm{sk}(e^{\hat{\xi}\theta})^\vee\| \le 1\right)$$

$$\Rightarrow \dot{E}_3 = -e_3^T N_3^T K_3 N_3 e_3 + p_{ee21}^T e^{\hat{\xi}\theta_{ec21}} v_{w1}^b + (\mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^\vee)^T \omega_{w1}^b + p_{ee32}^T e^{\hat{\xi}\theta_{ee32}} v_{w2}^b + (\mathrm{sk}(e^{\hat{\xi}\theta_{ee32}})^\vee)^T K_{21}\mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^\vee$$

Influence of Position Estimation Error p_{eeij} : Future Work

(A1, A4) $v_{wi}^b = 0,\ i \in \{1,2,3\},\ \omega_{w1}^b = 0$ Future Work

($\mathcal{L}_p$ -norm Performance ?)

$$\Rightarrow \dot{E}_3 = -e_3^T N_3^T K_3 N_3 e_3 + K_{21}(\mathrm{sk}(e^{\hat{\xi}\theta_{ee32}})^\vee)^T \mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^\vee$$

Convergence Analysis (Chain Type)

$$\begin{aligned}\dot{E}_3 = & -K_{21}\|e_{c21}\|^2 - K_{32}\|e_{c32}\|^2 - K_{e21}\|p_{ee21}\|^2 - K_{e32}\|p_{ee32}\|^2 \\ & -K_{e21}\|e_{ee21} - e_{c21}\|^2 - K_{e32}\|e_{ee32} - e_{c32}\|^2 + K_{21}e_{ee32}^T e_{c21}\end{aligned}$$

$\|a-b\|^2 = \|a\|^2 + \|b\|^2 - 2a^Tb$

$$\begin{aligned} = & -(K_{21}+K_{e21})\|e_{c21}\|^2 - (K_{32}+K_{e32})\|e_{c32}\|^2 - K_{e21}\|e_{ee21}\|^2 \\ & -K_{e32}\|e_{ee32}\|^2 - K_{e21}\|p_{ee21}\|^2 - K_{e32}\|p_{ee32}\|^2 \\ & +2K_{e21}e_{ee21}^T e_{c21} + 2K_{e32}e_{ee32}^T e_{c32} + K_{21}e_{ee32}^T e_{c21}\end{aligned}$$

$a^Tb \le \|a\|\ \|b\|$ (Conservative)

$$\begin{aligned} \le & -(K_{21}+K_{e21})\|e_{c21}\|^2 - (K_{32}+K_{e32})\|e_{c32}\|^2 - K_{e21}\|e_{ee21}\|^2 \\ & -K_{e32}\|e_{ee32}\|^2 - K_{e21}\|p_{ee21}\|^2 - K_{e32}\|p_{ee32}\|^2 + 2K_{e21}\|e_{ee21}\|\|e_{c21}\| \\ & +2K_{e32}\|e_{ee32}\|\|e_{c32}\| + K_{21}\|e_{ee32}\|\|e_{c21}\|\end{aligned}$$

$\|a\|\|b\| = \frac{1}{2}(\|a\|^2+\|b\|^2-(\|a\|-\|b\|)^2)$

$$\begin{aligned} = & -\frac{1}{2}K_{21}\|e_{c21}\|^2 - K_{32}\|e_{c32}\|^2 + \frac{1}{2}K_{21}\|e_{ee32}\|^2 - K_{e21}\|p_{ee21}\|^2 \\ & -K_{e32}\|p_{ee32}\|^2 - K_{e21}(\|e_{c21}\| - \|e_{ee21}\|)^2 \\ & -K_{e32}(\|e_{c32}\| - \|e_{ee32}\|)^2 - \frac{1}{2}K_{21}(\|e_{c21}\| - \|e_{ee32}\|)^2\end{aligned}$$

≥ 0

Convergence Analysis (Chain Type)

$$\Rightarrow \dot{E}_3 \le \psi_2 + \psi_3$$

$$\psi_2 := -\frac{1}{2}K_{21}\|e_{c21}\|^2 - K_{e21}\|p_{ee21}\|^2 - K_{e21}(\|e_{c21}\| - \|e_{ee21}\|)^2 \le 0$$

$$\begin{aligned}\psi_3 := & -K_{32}\|e_{c32}\|^2 + \frac{1}{2}K_{21}\|e_{ee32}\|^2 - K_{e32}\|p_{ee32}\|^2 \\ & -K_{e32}(\|e_{c32}\| - \|e_{ee32}\|)^2 - \frac{1}{2}K_{21}(\|e_{c21}\| - \|e_{ee32}\|)^2\end{aligned}$$

≥ 0

Lemma 2

If $K_{21} < \dfrac{2K_{32}K_{e32}}{K_{32}+K_{e32}}$, then $\psi_3 \le 0$. $\Rightarrow \dot{E}_3 \le 0$

(Proof) Completing square yields the following equation.

$$\begin{aligned}\psi_3 = & -(K_{32}+K_{e32})\left(\|e_{c32}\| - \frac{K_{e32}}{K_{32}+K_{e32}}\|e_{ee32}\|\right)^2 - K_{e32}\|p_{ee32}\|^2 \\ & -\frac{K_{32}K_{e32}}{K_{32}+K_{e32}}\left(\|e_{ee32}\| - \frac{K_{21}(K_{32}+K_{e32})}{2K_{32}K_{e32}}\|e_{c21}\|\right)^2 \\ & -\frac{K_{21}(2K_{32}K_{e32} - K_{21}(K_{32}+K_{e32}))}{4K_{32}K_{e32}}\|e_{c21}\|^2\end{aligned}$$

Convergence Analysis (Chain Type)

$$\begin{aligned}\dot{E}_3 \le & -\frac{1}{2}K_{21}\|e_{c21}\|^2 - K_{e21}\|p_{ee21}\|^2 - K_{e21}(\|e_{c21}\| - \|e_{ee21}\|)^2 \\ & -(K_{32}+K_{e32})\left(\|e_{c32}\| - \frac{K_{e32}}{K_{32}+K_{e32}}\|e_{ee32}\|\right)^2 - K_{e32}\|p_{ee32}\|^2 \\ & -\frac{K_{32}K_{e32}}{K_{32}+K_{e32}}\left(\|e_{ee32}\| - \frac{K_{21}(K_{32}+K_{e32})}{2K_{32}K_{e32}}\|e_{c21}\|\right)^2 \\ & -\frac{K_{21}(2K_{32}K_{e32} - K_{21}(K_{32}+K_{e32}))}{4K_{32}K_{e32}}\|e_{c21}\|^2\end{aligned}$$

≤ 0

Thus $\dot{E}_3 = 0 \to \lim_{t\to\infty} E_3 = 0.$

(Invariance Principle)

This means attitude synchronization in the sense of (2). □

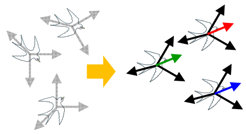


Fig. 6: Attitude Synchronization

Convergence Analysis (Chain Type)

n Rigid Bodies

$$\begin{bmatrix} \omega_{ec21}^b \\ \vdots \\ \omega_{ecn(n-1)}^b \\ V_{ee21}^b \\ \vdots \\ V_{een(n-1)}^b \end{bmatrix} = \begin{bmatrix} -e^{\hat{\xi}\theta_{ec21}} \cdots & 0 & 0 & I_3 & \cdots & 0 & 0 \\ \ddots & \vdots & \vdots & \ddots & & \vdots & \\ & -e^{\hat{\xi}\theta_{ecn(n-1)}} & 0 & 0 & & 0 & I_3 \\ & & -\mathrm{Ad}_{(g_{ee21}^{-1})} & & & 0 & \\ 0 & & & \ddots & & \vdots & \\ & & & & -\mathrm{Ad}_{(g_{een(n-1)}^{-1})} & & \end{bmatrix} \begin{bmatrix} \omega_{w2}^b \\ \vdots \\ \omega_{wn}^b \\ u_{e21} \\ \vdots \\ u_{en(n-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ V_{w1}^b \\ \vdots \\ V_{w(n-1)}^b \end{bmatrix}$$

Energy Function:

$$E_n := \sum_{i=1}^{n-1}\left(\phi(e^{\hat{\xi}\theta_{ec(i+1)i}}) + E(g_{ee(i+1)i})\right) \quad (8)$$

Differentiate (8) w.r.t. time

$$\begin{aligned}\dot{E}_n \le & -\sum_{i=1}^{n-1}\left(K_{(i+1)i}\|e_{c(i+1)i}\|^2 + K_{e(i+1)i}\left(\|p_{ee(i+1)i}\|^2 + (\|e_{c(i+1)i}\| - \|e_{ee(i+1)i}\|)^2\right)\right) \\ & +\frac{1}{2}\sum_{i=1}^{n-2}K_{(i+1)i}\left(\|e_{c(i+1)i}\|^2 + \|e_{ee(i+2)(i+1)}\|^2 - (\|e_{c(i+1)i}\| - \|e_{ee(i+2)(i+1)}\|)^2\right)\end{aligned}$$

Convergence Analysis (Chain Type)

$$\Rightarrow \dot{E}_n \le \sum_{i=2}^{n}\psi_i$$

$$\psi_2 := -\frac{1}{2}K_{21}\|e_{c21}\|^2 - K_{e21}\|p_{ee21}\|^2 - K_{e21}(\|e_{c21}\| - \|e_{ee21}\|)^2 \le 0$$

$$\begin{aligned}\psi_i := & -\frac{1}{2}K_{i(i-1)}\|e_{ci(i-1)}\|^2 + \frac{1}{2}K_{(i-1)(i-2)}\|e_{eei(i-1)}\|^2 \\ & -K_{ei(i-1)}\|p_{eei(i-1)}\|^2 - K_{ei(i-1)}(\|e_{ci(i-1)}\| - \|e_{eei(i-1)}\|)^2 \\ & -\frac{1}{2}K_{(i-1)(i-2)}(\|e_{c(i-1)(i-2)}\| - \|e_{eei(i-1)}\|)^2 \quad i = \{3,\cdots,n-1\}\end{aligned}$$

$$\begin{aligned}\psi_n := & -K_{n(n-1)}\|e_{cn(n-1)}\|^2 + \frac{1}{2}K_{(n-1)(n-2)}\|e_{een(n-1)}\|^2 \\ & -K_{en(n-1)}\|p_{een(n-1)}\|^2 - K_{en(n-1)}(\|e_{cn(n-1)}\| - \|e_{een(n-1)}\|)^2 \\ & -\frac{1}{2}K_{(n-1)(n-2)}(\|e_{c(n-1)(n-2)}\| - \|e_{een(n-1)}\|)^2\end{aligned}$$

From Lemma 2, $K_{(n-1)(n-2)} < \dfrac{2K_{n(n-1)}K_{en(n-1)}}{K_{n(n-1)} + K_{en(n-1)}}$ $\Rightarrow \psi_n \le 0$

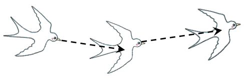
Convergence Analysis (Chain Type)

Lemma 3

If $K_{(i-1)(i-2)} < \dfrac{2K_{i(i-1)}K_{ei(i-1)}}{K_{i(i-1)} + 2K_{ei(i-1)}}$, then $\psi_i \le 0,\ i \in \{3, \cdots, n-1\}$.

(Proof) Completing square yields. (omit.)

Lemma 3 ➡ $\dot{E}_n \le \sum_{i=2}^{n} \psi_i \le 0$ $\cdots$



Chain Type

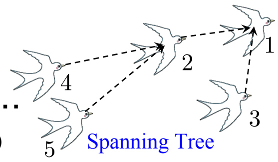
Thus $\dot{E}_n = 0 \rightarrow E_n = 0$.
(Invariance Principle)

This means attitude synchronization in the sense of (2). □

Convergence Analysis (Spanning Tree Type)

Assumption

(A6) Spanning Tree Type Visibility



Similar Approach
(Only take care of indices of rigid bodies)

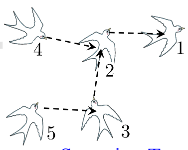
Theorem 2 (Visual Attitude Synchronization)
Consider the n rigid bodies with visual motion observer represented by (1) and (4). Then, under the assumptions A1, A2 and A4-A6, the control input (5), (6) achieves attitude synchronization in the sense of (2).

(Proof) Omit.

Extension: Desired Relative Attitude

Control Error: $e^{\hat{\xi}\theta_{ecij}} := e^{-\hat{\xi}\theta_{dij}} e^{\hat{\bar{\xi}}\bar{\theta}_{ij}}$

Simulation



Initial Condition

$p_{w1} = [5\ 0\ 5]^T$ $\quad \xi\theta_{w1}(0) = \left[0\ \frac{\pi}{4}\ 0\right]^T$

$p_{w2} = [0\ 0\ 0]^T$ $\quad \xi\theta_{w2}(0) = [0\ 0\ 0]^T$

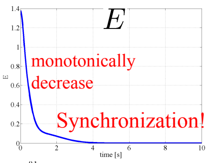
$p_{w3} = [0\ 0\ -5]^T$ $\quad \xi\theta_{w3}(0) = \left[0\ -\frac{\pi}{4}\ 0\right]^T$

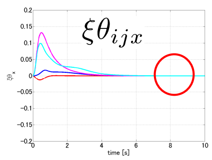
$p_{w4} = [-5\ 0\ 0]^T$ $\quad \xi\theta_{w4}(0) = \left[0\ \frac{\pi}{2}\ 0\right]^T$

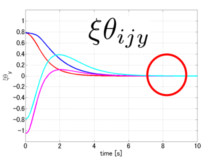
$p_{w5} = [-5\ 0\ -10]^T$ $\quad \xi\theta_{w5}(0) = [0\ 0\ 0]^T$

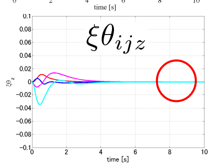
Gain

$K_{ij} = 3, K_{eij} = 5\ {}^{\forall} j \in \mathcal{N}_i, i \in \{2, \cdots, 5\}$









Outline

- Introduction
- Problem Setting
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  - Chain Type Visibility (Information Structure)
  - Spanning Tree Type Visibility
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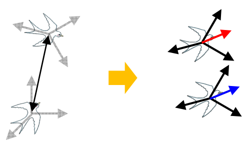
  - Mutual Visibility
- Future Works

Convergence Analysis (Mutual Visibility Type)

Mutual Visibility



2 Rigid Bodies

$$\begin{bmatrix} \omega^b_{ec12} \\ \omega^b_{ec21} \\ V^b_{ee12} \\ V^b_{ee21} \end{bmatrix} = \begin{bmatrix} -e^{\hat{\xi}\theta_{ec12}} & 0 & 0 \quad I_3 & 0 \quad 0 \\ 0 & -e^{\hat{\xi}\theta_{ec21}} & 0 \quad 0 & 0 \quad I_3 \\ 0 & 0 & -\mathrm{Ad}_{(g^{-1}_{ee12})} & 0 \\ 0 & 0 & 0 & -\mathrm{Ad}_{(g^{-1}_{ee21})} \end{bmatrix} \begin{bmatrix} \omega^b_{w1} \\ \omega^b_{w2} \\ u_{e12} \\ u_{e21} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ V^b_{w2} \\ V^b_{w1} \end{bmatrix}$$

Energy Function:

$$E_2 := \phi(e^{\hat{\xi}\theta_{ec12}}) + \phi(e^{\hat{\xi}\theta_{ec21}}) + E(g_{ee12}) + E(g_{ee21}) \quad (9)$$

Convergence Analysis (Mutual Visibility Type)

Differentiating (9) w.r.t. time

$$\begin{aligned} \dot{E}_2 &= \dot{E}(g_{ec12}) + \dot{E}(g_{ec21}) + \dot{E}(g_{ee12}) + E(g_{ee21}) \\ &= (\mathrm{sk}(e^{\hat{\xi}\theta_{ec12}})^{\vee})^T \omega^b_{ec12} + (\mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^{\vee})^T \omega^b_{ec21} \\ &\quad + e^T_{e12}\mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee12}})} V^b_{ee12} + e^T_{e21}\mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee21}})} V^b_{ee21} \\ &= -e^T_{c12}\omega^b_{w1} - e^T_{c21}\omega^b_{w2} - e^T_{e12}u_{e12} - e^T_{e21}u_{e21} + e^T_{c12}u_{we12} + e^T_{c21}u_{we21} \\ &\quad + e^T_{e12}\mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee12}})} V^b_{w2} + e^T_{e21}\mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee21}})} V^b_{w1} \\ &= -\underbrace{[e^T_{c12}\ e^T_{c21}\ e^T_{e12}\ e^T_{e21}]}_{=:\ e^T_2} \underbrace{\begin{bmatrix} I_3 & 0 & 0 & -I_3 & 0 \\ 0 & I_3 & 0 & 0 & -I_3 \\ 0 & 0 & I_6 & 0 & \\ 0 & 0 & 0 & I_6 & \end{bmatrix}}_{=:\ N^T_2} \underbrace{\begin{bmatrix} \omega^b_{w1} \\ \omega^b_{w2} \\ u_{e12} \\ u_{e21} \end{bmatrix}}_{=:\ u_2} \\ &\quad + e^T_{e12}\mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee12}})} V^b_{w2} + e^T_{e21}\mathrm{Ad}_{(e^{\hat{\xi}\theta_{ee21}})} V^b_{w1} \end{aligned}$$

Convergence Analysis (Mutual Visibility Type)

Control Input

$$u_2 = K_2 N_2 e_2 = \begin{bmatrix} K_{12}\mathrm{sk}(e^{\hat{\xi}\theta_{ec12}})^\vee \\ K_{21}\mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^\vee \\ K_{e12}e_{e12} - K_{e12}\begin{bmatrix} 0 \\ \mathrm{sk}(e^{\hat{\xi}\theta_{ec12}})^\vee \end{bmatrix} \\ K_{e21}e_{e21} - K_{e21}\begin{bmatrix} 0 \\ \mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^\vee \end{bmatrix} \end{bmatrix} \quad K_2 = \begin{bmatrix} K_{12}I_3 & 0 & 0 & 0 \\ 0 & K_{21}I_3 & 0 & 0 \\ 0 & 0 & K_{e12}I_6 & 0 \\ 0 & 0 & 0 & K_{e21}I_6 \end{bmatrix}$$

$$\dot{E}_2 = -e_2^T N_2^T K_2 N_2 e_2 + p_{ee12}^T e^{\hat{\xi}\theta_{ee12}} v_{w2}^b + (\mathrm{sk}(e^{\hat{\xi}\theta_{ec12}})^\vee)^T K_{21}\mathrm{sk}(e^{\hat{\xi}\theta_{ec21}})^\vee + p_{ee21}^T e^{\hat{\xi}\theta_{ee21}} v_{w1}^b + (\mathrm{sk}(e^{\hat{\xi}\theta_{ee21}})^\vee)^T K_{12}\mathrm{sk}(e^{\hat{\xi}\theta_{ec12}})^\vee$$

(A1) $v_{wi}^b = 0,\ i \in \{1,2\}$

$$\dot{E}_2 = -e_2^T N_2^T K_2 N_2 e_2 + K_{21}(\mathrm{sk}(e^{\hat{\xi}\theta_{cc12}})^\vee)^T \mathrm{sk}(e^{\hat{\xi}\theta_{cc21}})^\vee + K_{12}(\mathrm{sk}(e^{\hat{\xi}\theta_{cc21}})^\vee)^T \mathrm{sk}(e^{\hat{\xi}\theta_{cc12}})^\vee$$

⋮ Appendix

Convergence Analysis (Mutual Visibility Type)

$$\dot{E}_2 \le -\frac{1}{2}K_{12}\|e_{c12}\|^2 - \frac{1}{2}K_{21}\|e_{c21}\|^2 + \frac{1}{2}K_{21}\|e_{ee12}\|^2 + \frac{1}{2}K_{12}\|e_{ee21}\|^2 - K_{e12}(\|e_{c12}\| - \|e_{ee12}\|)^2 - K_{e21}(\|e_{c21}\| - \|e_{ee21}\|)^2 - \frac{1}{2}K_{12}(\|e_{c21}\| - \|e_{ee12}\|)^2 - \frac{1}{2}K_{21}(\|e_{c12}\| - \|e_{ee21}\|)^2$$

Candidate

$$\psi_{12} = -\frac{1}{2}K_{12}\|e_{c12}\|^2 + \frac{1}{2}K_{21}\|e_{ee12}\|^2 - K_{e12}(\|e_{c12}\| - \|e_{ee12}\|)^2$$

$$\psi_{21} = -\frac{1}{2}K_{21}\|e_{c21}\|^2 + \frac{1}{2}K_{12}\|e_{ee21}\|^2 - K_{e21}(\|e_{c21}\| - \|e_{ee21}\|)^2$$

$\psi_{12} \le 0$ ➡ $K_{12} > 4K_{21}$

$\psi_{21} \le 0$ ➡ $K_{21} > 4K_{12}$

Contradiction!

It is difficult to find the gain condition where E_2 uniquely decrease.

Convergence Analysis (Mutual Visibility Type)

Direct Calculation

$$\dot{E}_2 \le -\frac{1}{2}K_{12}\|e_{c12}\|^2 - \frac{1}{2}K_{21}\|e_{c21}\|^2 + \frac{1}{2}K_{21}\|e_{ee12}\|^2 + \frac{1}{2}K_{12}\|e_{ee21}\|^2 - K_{e12}(\|e_{c12}\| - \|e_{ee12}\|)^2 - K_{e21}(\|e_{c21}\| - \|e_{ee21}\|)^2 - \frac{1}{2}K_{12}(\|e_{c21}\| - \|e_{ee12}\|)^2 - \frac{1}{2}K_{21}(\|e_{c12}\| - \|e_{ee21}\|)^2$$

$$K_{21}\|e_{ee12}\|^2 + K_{12}\|e_{ee21}\|^2 < K_{12}\|e_{c12}\|^2 + K_{21}\|e_{c21}\|^2 + 2K_{e12}(\|e_{c12}\| - \|e_{ee12}\|)^2 + 2K_{e21}(\|e_{c21}\| - \|e_{ee21}\|)^2 + K_{12}(\|e_{c21}\| - \|e_{ee12}\|)^2 + K_{21}(\|e_{c12}\| - \|e_{ee21}\|)^2$$

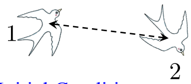
When $\begin{cases} \|e_{c12}\| \approx \|e_{c21}\| \approx \|e_{ee12}\| \approx \|e_{ee21}\| \\ \|e_{c12}\| > \|e_{ee12}\|, \|e_{c21}\| > \|e_{ee21}\| \end{cases}$, the condition is fragile

Relative Amplitude

The above condition depends on relative amplitude.

➡ Difficult to find boundedness like $\|e_{c12}\| < \gamma$ Future Work

Simulation



Mutual Visibility

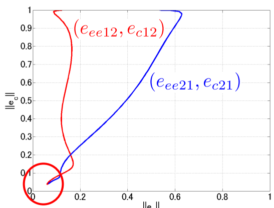
Initial Condition

$p_{w1} = [0\ 0\ 0]^T$ $\quad \xi\theta_{w1}(0) = \begin{bmatrix} 0 & -\frac{\pi}{6} & 0 \end{bmatrix}^T$

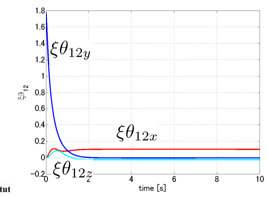
$p_{w2} = [5\ 0\ 0]^T$ $\quad \xi\theta_{w2}(0) = \begin{bmatrix} 0 & \frac{2\pi}{5} & 0 \end{bmatrix}^T$

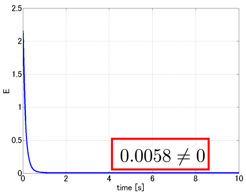
Gain

$K_{ij} = 3, K_{eij} = 5\ {}^\forall j \in \mathcal{N}_i, i \in \{1,2\}$



Want to find boundedness!





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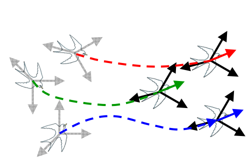
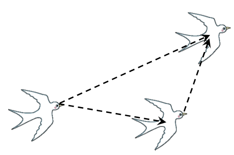
Future Work

Future Work

- Experiment
- Linear Velocity, Lead Rigid Body's Velocity ($\mathcal{L}_p$ -norm Performance)
- Boundedness Analysis of Mutual Visibility
- Convergence Analysis of Multiple Visibility $\omega_{wi}^b = \sum_{j \in \mathcal{N}_i} K_{ij}\mathrm{sk}(e^{\hat{\xi}\theta_{ij}})^\vee$



(Linear Velocity)



(Multiple Visibility)