Introduction to Team Theory Revisited

FL08-05-2
Tatsuya Miyano

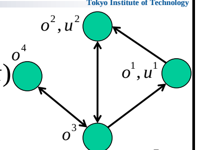
Outline

1. Introduction
2. State Feedback Control Law
3. Kalman Filtering
4. Output Feedback Control Law
5. Distributed Synthesis Procedure
6. Conclusion and Future Works

Discrete Time Linear Stochastic System

Distributed Control [1)]

$$\Sigma \begin{cases} x(t+1) = Ax(t) + Bu(t) + Ww(t) \\ o(t) = Fx(t) + Gu(t) + v(t) \end{cases}$$



$u(t) = [u^1(t), u^2(t), \ldots, u^m(t)]^T$, $o(t) = [o^1(t), o^2(t), \ldots, o^p(t)]^T$,

$x(0) = x_0, t = 0,1,\ldots,T.$ m control stations

$O_t = \{(\tau, k) \mid \tau = 0,1,\ldots,t; k = 1,2,\ldots,p\}$ p observation posts

$y^i(t) = \{o^k(\tau) \mid (\tau, k) \in Y_t^i \subset O_t\}$

observation data utilized by control station i

$$u^i(t) = \gamma^i(t, y^i(t))$$

Distributed Control Problem

The **common goal** for all control stations is as follows.
team

Problem 1 [1)]

$$\min_{\mathcal{U}} E\left\{ \sum_{t=1}^{T} (x(t+1)^T Q x(t+1) + u(t)^T R u(t)) \right\}$$

$Q \geq 0$, $R > 0$; E: expectation

subject to

$$\mathcal{U} = \{u^i(t) = \gamma^i(t, y^i(t)) \mid t = 0,1,\ldots,T; i = 1,2,\ldots,m\}$$

Remark 1 [1)]

$Y_t^i = O_t$ ⇒ classical information structure

$Y_t^i \neq O_t$ ⇒ nonclassical information structure

LQG with Nonclassical Information Structure

Remark 2 [1)]

(A) If information structure is nonclassical, then optimal solution of LQG control problem $\gamma^{opt} \in S$ is not always an affine function.

(B) If we restrict that $\gamma \in S$ is an affine function and information structure is nonclassical, then LQG control problem is in general nonconvex optimization problem.

⇓

We consider the special case.

Static Team Problem

Definition 1 [1)]

If no element of $\hat{u} = \{u^j(\tau) \mid j = 1,2,\ldots,m; \tau = 1,2,\ldots,t\}$ affects $y^i(t)$, then we call the **distributed control problem** the **static team problem**.

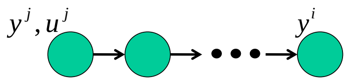
Theorem 1 [1)]

Static team problem is convex optimization problem and the only optimal solution is an affine function.

Partially Nested Information Structure

Definition 2 [1), 2)]

If what u^j affects y^i implies $y^j \subset y^i$ for all i, j, then the information structure is called **partially nested**.



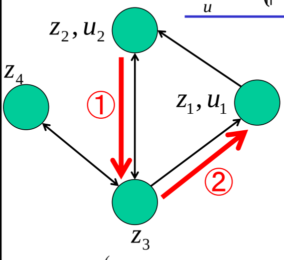
Theorem 2 [1)]

If the information structure is **partially nested**, then the **distributed control problem** is equivalent to the **static team problem**.

Distributed Control by Covariance Constraints

Problem 2 [6), 7)]

$$\min_u E\left\{\|z\|^2 + \|u\|^2\right\}$$ stationary variance



$$\begin{bmatrix} z_1(t+1) \\ z_2(t+1) \\ z_3(t+1) \\ z_4(t+1) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & 0 & \Phi_{13} & 0 \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & 0 \\ 0 & \Phi_{32} & \Phi_{33} & \Phi_{34} \\ 0 & 0 & \Phi_{43} & \Phi_{44} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix} + \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{bmatrix}$$

$$u_1(t) = \mu_1\left(\bar{z}_1(t), \bar{z}_2(t-2), \bar{z}_3(t-1), \bar{z}_4(t-2)\right)$$
$$u_2(t) = \mu_2\left(\bar{z}_1(t-1), \bar{z}_2(t), \bar{z}_3(t-1), \bar{z}_4(t-2)\right)$$
$$\bar{z}_i(t) = \begin{bmatrix} z_i(t) \\ z_i(t-1) \\ z_i(t-2) \\ \vdots \end{bmatrix}$$

Distributed Control by Covariance Constraints

$$u_1(t) = \mu_1\left(\bar{z}_1(t), \bar{z}_2(t-2), \bar{z}_3(t-1), \bar{z}_4(t-2)\right)$$
$$u_2(t) = \mu_2\left(\bar{z}_1(t-1), \bar{z}_2(t), \bar{z}_3(t-1), \bar{z}_4(t-2)\right)$$

⇓ $\bar{z}(t-2)$ is replaced by $z(t-2)$.

$$u_1(t) = \bar{\mu}_1\left(z(t-2), w_1(t-2), w_1(t-1), w_3(t-2)\right)$$
$$u_2(t) = \bar{\mu}_2\left(z(t-2), w_1(t-2), w_2(t-2), w_2(t-1), w_3(t-2)\right)$$

$$z(t+1) = \Phi z(t) + \Gamma u(t) + w(t)$$

⇓ Time delay is at most 2-steps.

$$x(t+1) = Ax(t) + Bu(t) + Ww(t) \quad \cdots (1)$$

$$x(t) = \begin{bmatrix} z(t)^T & w(t-1)^T & w(t-2)^T \end{bmatrix}^T = \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_{12}(t) \end{bmatrix}^T$$

Distributed Control by Covariance Constraints

The **covariance constraints** are as follows.

$$\gamma \geq E\left(\|z\|^2 + \|u\|^2\right) = E\left(\|x\|^2 + \|u\|^2\right) - 8$$
$$0 = Ew_2(t-1)u_1(t) = Ex_6(t)u_1(t)$$
$$0 = Ew_2(t-2)u_1(t) = Ex_{10}(t)u_1(t)$$
$$0 = Ew_3(t-1)u_1(t) = Ex_7(t)u_1(t)$$
$$0 = Ew_4(t-1)u_1(t) = Ex_8(t)u_1(t)$$
$$0 = Ew_4(t-2)u_1(t) = Ex_{12}(t)u_1(t)$$
$$0 = Ew_1(t-1)u_2(t) = Ex_5(t)u_2(t)$$
$$0 = Ew_3(t-1)u_2(t) = Ex_7(t)u_2(t)$$
$$0 = Ew_4(t-1)u_2(t) = Ex_8(t)u_2(t)$$
$$0 = Ew_4(t-2)u_2(t) = Ex_{12}(t)u_2(t)$$

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State Feedback with Covariance Constraints

Theorem 3 [6), 7)] For every $\bar{\gamma}$, the following statements are equivalent.

(i) There exists a feedback law $u(t) = \mu(x(t))$ that together with (1) has a stationary zero mean solution satisfying **covariance constraints**

$$E\begin{bmatrix} x \\ u \end{bmatrix}^T Q^1 \begin{bmatrix} x \\ u \end{bmatrix} = E\begin{bmatrix} x \\ u \end{bmatrix}^T Q^2 \begin{bmatrix} x \\ u \end{bmatrix} = \ldots = E\begin{bmatrix} x \\ u \end{bmatrix}^T Q^J \begin{bmatrix} x \\ u \end{bmatrix} \leq \bar{\gamma}.$$

(ii) There exists a positive semidefinite $X = E\begin{bmatrix} x \\ u \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}^T = \begin{bmatrix} X_{xx} & X_{xu} \\ X_{ux} & X_{uu} \end{bmatrix}$ with

$$X_{xx} \geq \begin{bmatrix} A & B \end{bmatrix} X \begin{bmatrix} A & B \end{bmatrix}^T + WW^T \quad \cdots (2)$$
$$\bar{\gamma} \geq \mathrm{tr}(XQ^1) = \mathrm{tr}(XQ^2) = \ldots = \mathrm{tr}(XQ^J) \quad \cdots (3)$$

State Feedback with Covariance Constraints

Moreover, if $L = X_{ux}X_{xx}^{-1}$ and X satisfies the conditions of (ii), then the conditions of (i) hold for the linear control law $u = Lx + \nu$, where ν is a zero mean stochastic variable independent of w and x and with $E\nu\nu^T = X_{uu} - X_{ux}X_{xx}^{-1}X_{xu}$.

Remark 3 [12)]

$$E\begin{bmatrix}x\\u\end{bmatrix}^T Q^i \begin{bmatrix}x\\u\end{bmatrix} \le \bar{\gamma} \Leftrightarrow \mathrm{tr}\left(XQ^i\right) \le \bar{\gamma}$$

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Kalman Filtering with Uncertain Covariance

Theorem 4 [7)] For every $\bar{\gamma}$, (i) and (ii) are equivalent.

$$\begin{bmatrix}x^j(t+1)\\y(t)\end{bmatrix} = \begin{bmatrix}Ax^j(t)+v^j(t)\\Cx^j(t)+e^j(t)\end{bmatrix},\ E\begin{bmatrix}v^j(t)\\e^j(t)\end{bmatrix}\begin{bmatrix}v^j(t)\\e^j(t)\end{bmatrix}^T = R^j \quad j = 1,2,\dots,J$$

(i) There exists a map ν such that the state estimate $\hat{x}(t) = \nu(y(t-1), y(t-2), y(t-3),\dots)$ for all t satisfies

covariance constraints

$$E\left\|x^1(t)-\hat{x}(t)\right\|^2 = E\left\|x^2(t)-\hat{x}(t)\right\|^2 = \dots = E\left\|x^J(t)-\hat{x}(t)\right\|^2 \le \bar{\gamma}.$$

(ii) There exists a positive semidefinite $S = \begin{bmatrix}S_{xx} & S_{xy}\\S_{yx} & S_{yy}\end{bmatrix}$ with

$$S_{xx} \ge [A \quad C]S[A \quad C]^T + I \quad \cdots(4)$$

$$\bar{\gamma} \ge \mathrm{tr}(SR^1) = \mathrm{tr}(SR^2) = \dots = \mathrm{tr}(SR^J). \quad \cdots(5)$$

Kalman Filtering with Uncertain Covariance

Moreover, if $K = S_{xx}^{-1}S_{xy}$ and S satisfies the conditions in (ii), then the conditions of (i) hold for the estimator defined by

$$\hat{x}(t+1) = A\hat{x}(t) + K\left[C\hat{x}(t) - y(t)\right].$$

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Output Feedback with Covariance Constraints

Theorem 5 [7)] For every γ, (i) and (ii) are equivalent.

$$\begin{bmatrix}x^j(t+1)\\y(t)\end{bmatrix} = \begin{bmatrix}Ax^j(t)+Bu(t)+v^j(t)\\Cx^j(t)+e^j(t)\end{bmatrix},\ E\begin{bmatrix}v^j(t)\\e^j(t)\end{bmatrix}\begin{bmatrix}v^j(t)\\e^j(t)\end{bmatrix}^T = R^j \quad j = 1,2,\dots,J$$

(i) There exists a stabilizing feedback law $u(t) = \nu(y(t-1), y(t-2), y(t-3),\dots)$ with stationary solutions for $j = 1,2,\dots,J$ such that

$$E\begin{bmatrix}x^j\\u\end{bmatrix}^T Q^i \begin{bmatrix}x^j\\u\end{bmatrix}$$

covariance constraints

has the same value for all $i, j \in \{1,2,\dots,J\}$ and the value is not greater than γ.

Output Feedback with Covariance Constraints

(ii) The conditions of (i) hold for the feedback law defined by

$$\hat{x}(t+1) = A\hat{x}(t) + S_{xx}^{-1}S_{xy}\left[C\hat{x}(t) - y(t)\right]$$

$$u(t) = X_{ux}X_{xx}^{-1}\hat{x}(t)$$

where the matrix $X = \begin{bmatrix} X_{xx} & X_{xu} \\ X_{ux} & X_{uu} \end{bmatrix}$ satisfies (2), (3) with minimal possible $\bar{\gamma}$ and $S = \begin{bmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{bmatrix}$ satisfies (10), (11) with minimal possible $\hat{\gamma}$.

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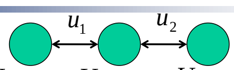
Dual Decomposition



U_1 U_2 U_3 U_i : concave

$$\max_{u_1,u_2} U_1(w_1+u_1) + U_2(w_2-u_1+u_2) + U_3(w_3-u_2)$$

dual decomposition

$$\min_{\lambda_{21},\lambda_{23}} \max_{u_{11},u_{21},u_{22},u_{32}} U_1(w_1+u_{11}) + U_2(w_2-u_{21}+u_{22})$$
$$+U_3(w_3-u_{32}) + \lambda_{21}(u_{21}-u_{11}) + \lambda_{23}(u_{32}-u_{22})$$

for fixed λ_{jk}

$$\max_{u_{11}} U_1(w_1+u_{11}) - \lambda_{21}u_{11}$$

$$\max_{u_{21},u_{22}} U_2(w_2-u_{21}+u_{22}) + \lambda_{21}u_{21} - \lambda_{23}u_{22}$$

$$\max_{u_{32}} U_3(w_3-u_{32}) + \lambda_{23}u_{32}$$

Saddle Point Algorithm

Algorithm 1 [8)]

A gradient search for the saddle point

$$\min_{\lambda}\max_{x} U(\lambda, x)$$

has the dynamics

$$\dot{\lambda} = -\frac{\partial U}{\partial \lambda}, \quad \dot{x} = \frac{\partial U}{\partial x}.$$

U : strictly convex-concave

Static Team Problem

Theorem 6 [8)]

$$\min_{K_1,K_2,\ldots,K_J} E\sum_{j=1}^{J}\left\| v_j + \sum_{k=1}^{J} B_{jk}(K_k y_k)\right\|_{Q_j}^2$$

$$= \max_{(\lambda_{jk})}\min_{(K_{jk})} E\sum_{j=1}^{J}\left(\left\| v_j + \sum_{k=1}^{J} B_{jk}(K_k y_k)\right\|_{Q_j}^2 + \sum_{k=1}^{J}\left(\lambda_{kj}^{*}K_{jj}y_j - \lambda_{jk}^{*}K_{jk}y_k\right)\right)$$

Dynamic Team Problem

Theorem 7 [8)]

$$\min_{K_1,K_2,\ldots,K_J} \sum_{j,k,l}\int_{-\pi}^{\pi} \mathrm{tr}\left(Q_j\left[I_{jk} \quad B_{jk}K_k(e^{i\omega})\right]\phi_{kl}(\omega)\left[I_{jl} \quad B_{jl}K_l(e^{i\omega})\right]^{*}\right)d\omega$$

$$= \max_{(\Lambda_{jk})}\min_{(K_{jk})} \sum_{j,k,l}\int_{-\pi}^{\pi} \left\{\mathrm{tr}\left(Q_j\left[I_{jk} \quad B_{jk}K_k(e^{i\omega})\right]\phi_{kl}(\omega)\left[I_{jl} \quad B_{jl}K_l(e^{i\omega})\right]^{*}\right)\right.$$
$$\left. + \mathrm{tr}\left[\Lambda_{jk}(e^{i\omega})^{*}K_{jk}(e^{i\omega})\right] - \mathrm{tr}\left[\Lambda_{kj}(e^{i\omega})^{*}K_{jj}(e^{i\omega})\right]\right\}d\omega$$

Remark 4 [8)]

$$E\sum_{j=1}^{J}\left\| v_j + \sum_{k=1}^{J} B_{jk}(K_k * y_k)\right\|_{Q_j}^2$$

$$\Leftrightarrow \sum_{j,k,l}\int_{-\pi}^{\pi} \mathrm{tr}\left(Q_j\left[I_{jk} \quad B_{jk}K_k(e^{i\omega})\right]\phi_{kl}(\omega)\left[I_{jl} \quad B_{jl}K_l(e^{i\omega})\right]^{*}\right)d\omega$$

Outline

Conclusion and Future Works

Conclusion

- We have introduced team theory.

Future Works

- A distributed algorithm with information structures considering complexity
- An expansion to predictive control
- A graph theoretical approach to information structures

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Distributed Control by Covariance Constraints

Problem 3 [12)]

$$\min_{u} Ex^T(N)Q_{xx}x(N) + \sum_{k=0}^{N-1} E\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

State Feedback with Covariance Constraints

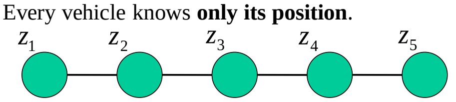
Example 2: Vehicle Formation Control[6)]

$$\min_{u_1,u_2,\ldots,u_5} E(z_1)^2 + E\sum_{i=1}^{4}(z_i - z_{i+1})^2 + E(z_5)^2 + 10E\sum_{i=1}^{5}(u_i)^2$$

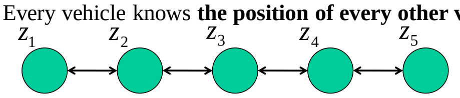
$$z_i(t+1) = z_i(t) + u_i(t) + w_i(t)$$

$$i = 1,2,\ldots,5$$

1) Every vehicle knows **only its position**.

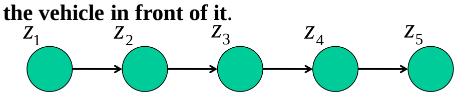


2) Every vehicle knows **the position of every other vehicle**.

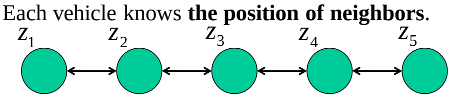


State Feedback with Covariance Constraints

3) Each vehicle knows **its position and also the position of the vehicle in front of it**.



4) Each vehicle knows **the position of neighbors**.



optimal value of the cost function

$$27.91^{1)} > 27.27^{3)} > 26.53^{4)} > 26.49^{2)}$$

Kalman Filtering

Definition of LQG[11)]

1) Linearity
2) Whiteness
3) Gaussian
4) Quadratic Criteria