Output Synchronization in SE(3) -Passivity Approach-

Output Synchronization in SE(3)

– Passivity Approach -

SE(3)における出力協調
- 受動性アプローチ-

FL07-14-2

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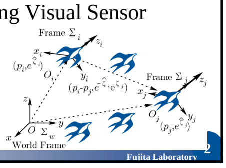
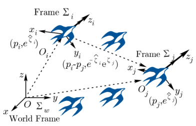
Outline



Previous Results

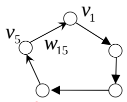
- **Agent Model** $(i=1,\cdots,n)$

$$\dot p_i = e^{\hat\xi_i} v_i \qquad p_i \in \mathcal{R}^3 \quad \text{position}$$
$$\varsigma_i = e^{\hat\xi_i}\omega_i^b \quad (1) \qquad e^{\hat\xi_i}\in SO(3) \quad \text{orientation}$$
$$\varsigma_i = \theta_i\xi_i \qquad v_i\in\mathcal{R}^3 \quad \text{body velocity}$$
$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}^{\wedge} := \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix} \qquad \omega_i^b\in\mathcal{R}^3 \quad \text{angular velocity}$$
$$\theta_i\in\mathcal{R} \quad \text{rotation angle} \qquad \xi_i\in\mathcal{R}^3 \quad \text{rotation axes}$$



- **Using a graph to represent the Intersection topology**

Graph G : Graph consist of a pair $(V(G),E(G),W(G))$, where $V(G)$ is a finite nonempty set of nodes, $E(G)\subseteq V(G)\times V(G)$ is a set of pair of nodes ,called edges and W(G) is a set of weights over the set of edges.



$G:=(V,E)$: Graph
$V:=\{1,\cdots n\}$: A set of vertices indexed by set of agents
$E\subseteq V\times V$: A set of edges the represent the neighboring relations
W : A set of weights over the set of edges
neighborhood N_i : A set of agents whose information is available to agent i

Previous Results

- **Goal** Attitude Coordination

$$\lim_{t\to\infty}(e^{\hat\xi_i(t)} - e^{\hat\xi_j(t)}) = 0$$

- **Control Input** $\omega_i^b = \sum_{j\in N_i} w_{ij}\,\mathrm{sk}(e^{-\hat\xi_i}e^{\hat\xi_j})^{\vee}$ $\quad w_{ij}$ is weight of edge

- **Analysis** (07 5/15 FL semi)

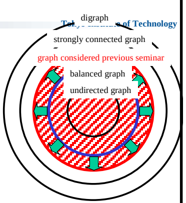
Assumptions
- (A1) At the initial time $t=0$, the agents' orientation matrices, $e^{\hat\xi_i(0)}\forall i$ are positive definite
- (A2) $|v_i|=1\ \forall i$ each agent's speed is constant and normalized.
- (A3)Graph is fixed, strongly connected and each weights are positive. i.e. $w_{ij}>0\quad\forall i,j$
- (A4) $\gamma^T L_w = 0^T\quad \gamma^T=[\gamma_1\ \cdots\ \gamma_n]\quad \gamma_i>0\quad\forall i$

L_w : Weighted Graph Laplacian

Potential Function

$$V=\sum_{i=1}^n \gamma_i\phi(e^{\hat\xi_i}) := \sum_{i=1}^n \frac{1}{2}\gamma_i \mathrm{tr}(I_3 - e^{\hat\xi_i}) \qquad L_w=\{L_{wij}\}:=\begin{cases}\sum_{j\in N_i} w_{ij} & \text{if } i=j\\ -w_{ij} & \text{if } j\in N_i\\ 0 & \text{if } j\notin N_i\end{cases}$$

Outline



Remark on Previous Results

- Remark on Previous Result

In this my presentation, we first show that the assumption (A4)

- $\gamma^T L_w = 0^T\quad \gamma^T=[\gamma_1\ \cdots\ \gamma_n]\quad \gamma_i>0\quad\forall i$

L_w : Weighted Graph Laplacian

is satisfied if the graph is strongly connected.

assumption (A3)

This means the assumption (A4) is redundant.

We prove the following lemma.

lemma1 Consider the weighted graph and each weights are positive, i.e. $w_{ij}>0$ $\forall i,j$. If the weighted graph is strongly connected, there exists a positive vector γ such that $L_w^T\gamma=0$, where L_w : is weighted graph laplacian defined by

$$L_w=\{L_{wij}\}:=\begin{cases}\sum_{j\in N_i} w_{ij} & \text{if } i=j\\ -w_{ij} & \text{if } j\in N_i\\ 0 & \text{if } j\notin N_i\end{cases}.$$

Output Synchronization in SE(3) -Passivity Approach-

Reducible and Irreducible

In order to prove lemma1, we use the following things.

- reducible(可約性) and Irreducible (既約性)
- nonnegative matrices (非負行列) and Perron-Frobenius theorems

Definition ([1], pp. 361 6.2.21 Definition, [2], pp.11) reducible

A matrix $A \in \mathcal{R}^{n \times n}$ is said to be **reducible** if either

(a) $n = 1$ and $A = 0$; or

(b) $n \geq 2$, there is a permutation matrix $P \in \mathcal{R}^{n \times n}$, and there is some integer r with $1 \leq r \leq n-1$ such that

$$P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where $B \in \mathcal{R}^{r \times r}, D \in \mathcal{R}^{n-r \times n-r}, C \in \mathcal{R}^{r \times n-r}$ and $0 \in \mathcal{R}^{n-r \times r}$ is a zero matrix.

Definition ([1], pp. 362 6.2.22 Definition, [2], pp.11) irreducible

A matrix $A \in \mathcal{R}^{n \times n}$ is said to be **irreducible** if it is not reducible.

[1] R. Horn and C. Johnson, ''*Matrix Analysis*'', 1985
[2] 児玉、須田: システム制御のためのマトリクス理論, 1978

Irreducible and strongly connected

Example

$$A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ is reducible. } A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ is irreducible.}$$

In fact, there is relationship between irreducible matrix and strongly connected graph.

Theorem ([1], pp. 362 6.2.24 Theorem, [2], pp. 24 定理 1.2)

Let $A \in \mathcal{R}^{n \times n}$. The following are equivalent:

(a) A is irreducible;

(b) $\Gamma(A)$ is strongly connected.

(* $\Gamma(A)$ means the graph having the adjacency matrix A.)

From definition and theorem, If matrix $A \in \mathcal{R}^{n \times n}$ is irreducible, then $-A$ and weighted graph laplacian L_w are irreducible.

[1] R. Horn and C. Johnson, ''*Matrix Analysis*'', 1985
[2] 児玉、須田: システム制御のためのマトリクス理論, 1978

Nonnegative Matrix and Perron-Frobenius theorems

Definition ([1], pp. 359 6.2.17 Definition, [2], pp.295) nonnegative matrix

Let $A = [a_{ij}] \in \mathcal{R}^{n \times n}$. We say that A is nonnegative if all its entries a_{ij} are nonnegative. We say that A is positive if all its entries a_{ij} are positive.

Example

$$A_2 = \begin{bmatrix} 2 & 10 & 1 \\ 10 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \text{ is nonnegative matrix. } A_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is not nonnegative matrix.}$$

Theorem Perron-Frobenius Theorems ([1], pp. 508 8.4.4 Theorem, [2], pp. 304 定理11.4)

Let $A \in \mathcal{R}^{n \times n}$ and suppose that A is irreducible and nonnegative. Then

(a) $\rho(A) > 0$; (* $\rho(A)$ is the spectral radius.)

(b) $\rho(A)$ is eigenvalue of A;

(c) There is a positive vector x such that $Ax = \rho(A)x$; and

(d) $\rho(A)$ is an algebraically (and hence geometrically) simple eigenvalue of A.

(* Positive vector is a vector whose elements are positive.)

[1] R. Horn and C. Johnson, ''*Matrix Analysis*'', 1985
[2] 児玉、須田: システム制御のためのマトリクス理論, 1978

Lemma1

We try to prove the following lemma

lemma1 Consider the weighted graph and each weights are positive, i.e. $w_{ij} > 0$ $\forall i, j$. If the weighted graph is strongly connected, there exists a positive vector γ such that $L_w^T \gamma = 0$, where L_w : is weighted graph laplacian defined by

$$L_w = \{L_{wij}\} := \begin{cases} \sum_{j \in N_i} w_{ij} & \text{if } i = j \\ -w_{ij} & \text{if } j \in N_i \\ 0 & \text{if } j \notin N_i \end{cases}.$$

Proof:

Define the weighted graph laplacian as the following

$$L_w = \{L_{wij}\} := \begin{cases} \sum_{j \in N_i} w_{ij} & \text{if } i = j \\ -w_{ij} & \text{if } j \in N_i \\ 0 & \text{if } j \notin N_i \end{cases}$$

and matrix P as the following

$$P := \max_i L_{wii} I - L_w^T. \quad (1)$$

$\max_i L_{wii}$ is the maximum diagonal element of matrix L_w.

I is elementary matrix.

Lemma1

If the graph is strongly connected and each weights are positive, the matrix P has the following properties.

(a) the matrix P is irreducible.

(b) the matrix P is nonnegative.

Using Perron-Frobenius theorems, there is positive vector γ such that

$$P\gamma = \rho(P)\gamma. \quad (2)$$

$\rho(P)$ is spectral radius of matrix P.

From Definition matrix P, the following equation is satisfied

$$\lambda(P) = \max_i L_{wii} - \lambda(L_w)$$

where $\lambda(P)$ is eigenvalue of matrix P and $\lambda(L_w)$ is eigenvalue of matrix L_w.

So we can show

$$\rho(P) = \max \lambda(P) = \max_i L_{wii}. \quad (3)$$

Substitute the equation (1) and (3) to the equation (2)

$$\left(\max_i L_{wii} - L_w^T\right)\gamma = \left(\max_i L_{wii}\right)\gamma$$

$$-L_w^T \gamma = 0$$

$$L_w^T \gamma = 0 \quad \gamma \text{ is a positive vector.}$$

Q. E. D.

Attitude Coordination in SE(3)

- **Agent Model** $(i = 1, \cdots, n)$

$$\dot{p}_i = e^{\hat{\xi}_i} v_i, \quad \varsigma_i = e^{\hat{\xi}_i} \omega_i^b, \quad \varsigma_i = \theta_i \xi_i \quad (1)$$

$p_i \in \mathcal{R}^3$ position

$e^{\hat{\xi}_i} \in SO(3)$ orientation

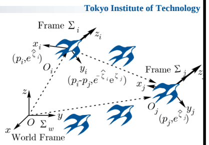
$v_i \in \mathcal{R}^3$ body velocity

$\omega_i^b \in \mathcal{R}^3$ angular velocity

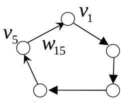
$\theta_i \in \mathcal{R}$ rotation angle

$\xi_i \in \mathcal{R}^3$ rotation axes

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}^\wedge := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$



- **Using a graph to represent the Intersection topology**



Graph G : Graph consist of a pair $(V(G), E(G), W(G))$, where $V(G)$ is a finite nonempty set of nodes, $E(G) \subseteq V(G) \times V(G)$ is a set of pair of nodes ,called edges and W(G) is a set of weights over the set of edges.

$G := (V, E)$: Graph

$V := \{1, \cdots, n\}$: A set of vertices indexed by set of agents

$E \subseteq V \times V$: A set of edges the represent the neighboring relations

W : A set of weights over the set of edges

neighborhood N_i : A set of agents whose information is available to agent i

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Attitude Coordination in SE(3)

- **Goal** Attitude Coordination

$$\lim_{t\to\infty}(e^{\hat{\xi}_i(t)} - e^{\hat{\xi}_j(t)}) = 0$$

- **Control Input** $\omega_i^b = k_i \sum_{j\in N_i} w_{ij}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$ w_{ij} is weight of edge

- **Analysis**

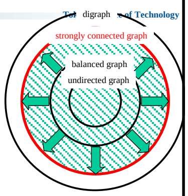
Assumptions
- (A1) At the initial time $t=0$, the agents' orientation matrices, $e^{\hat{\xi}_i(0)}\forall i$ are positive definite
- (A2) $|v_i| = 1\ \forall i$ each agent's speed is constant and normalized.
- (A3)Graph is fixed, strongly connected and each weights are positive. i.e. $w_{ij} > 0\ \ \forall i, j$
- ~~(A4) $\gamma^T L_w = 0^T$ $\gamma^T = [\gamma_1 \ \cdots \ \gamma_n]$ $\gamma_i > 0 \ \ \forall i$~~

Potential Function L_w : Weighted Graph Laplacian

$$V = \sum_{i=1}^n \frac{\gamma_i}{k_i}\phi(e^{\hat{\xi}_i}) := \sum_{i=1}^n \frac{1}{2}\frac{\gamma_i}{k_i}\mathrm{tr}(I_3 - e^{\hat{\xi}_i}) \qquad L_w = \{L_{wij}\} := \begin{cases}\sum_{j\in N_i} w_{ij} & \text{if } i=j\\ -w_{ij} & \text{if } j\in N_i\\ 0 & \text{if } j\notin N_i\end{cases}$$

$$\gamma^T L_w = 0^T \qquad \gamma^T = [\gamma_1 \ \cdots \ \gamma_n] \ \ \gamma_i > 0 \ \ \forall i$$

Outline



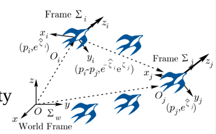
Problem Statement

- **Rigid Body Motion** $(i = 1, \cdots, n)$

$$\dot{p}_i = e^{\hat{\xi}_i}v_i^b \quad \dot{e}^{\hat{\xi}_i} = e^{\hat{\xi}_i}\hat{\omega}_i^b \quad (1) \quad \varsigma_i = \theta_i\xi_i \quad y_i = [p_i \ \ e^{\hat{\xi}_i}]$$

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}^\wedge := \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}$$

$p_i \in \mathcal{R}^3$ position
$e^{\hat{\xi}_i} \in SO(3)$ orientation
$v_i^b \in \mathcal{R}^3$ body velocity
$\omega_i^b \in \mathcal{R}^3$ angular velocity
$\theta_i \in \mathcal{R}$ rotation angle
$\xi_i \in \mathcal{R}^3$ rotation axes
$y_i \in SE(3)$ output



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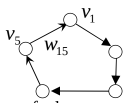

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neighborhood N_i : A set of agents whose information is available to agent i

Goal

- **Goal** Output Synchronization ([3], pp. 109 Definition1)

In the absence of communication delays, the agents are said to output synchronize if

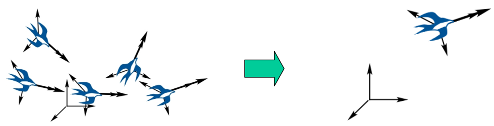
$$\lim_{t\to\infty}|y_i(t) - y_j(t)| = 0 \qquad \forall i, j = 1, \cdots, n$$

A group of agents is said to output synchronization,when all agents converge to the same output between the agents.

In reference [1] and [2], $y_i(t)\in R^m$. In my case, $y_i(t)\in SE(3)$.

This is a different point between reference [1] and my case.

In this case, $y_i = [p_i \ \ e^{\hat{\xi}_i}]$. So, if output synchronization is achieved, all agents have the same position and orientation.



[3] N. Chopra and M. W. Spong, ''Passivity-Based Control of Multi-Agent Systems'' in Advances in Robot Control: From Everyday Physics to Human-Like Movements, pp. 107-134, Springer-Verlag, Berlin, 2006

Control Input

- **Control Input**

$$\begin{bmatrix}v_i^b\\ \omega_i^b\end{bmatrix} = \sum_{j\in N_i} w_{ij}\begin{bmatrix}k_{pi}I & 0\\ 0 & k_{ei}I\end{bmatrix}\begin{bmatrix}e^{-\hat{\xi}_i} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}p_j - p_i\\ \mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee\end{bmatrix} \quad (2)$$

$k_{pi} > 0$ $k_{ei} > 0$ $\forall i = \{1, \cdots, n\}$

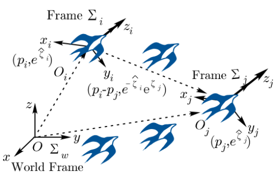
Control Input in Σ_i · Weight · Gain · Translate matrix $\Sigma_w \to \Sigma_i$ · Output error in Σ_w

$p_j - p_i$: Position error
$e^{-\hat{\xi}_i}e^{\hat{\xi}_j}$: Relative orientation
N_i : Agent i's neighborhood

$$\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}^\vee = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$$

Attitude Coordination

$$\omega_i^b = \sum_{j\in N_i} k_i w_{ij}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$$



Assumptions

- **Assumptions** (A)
  - (A1) At the initial time $t=0$, the agents' orientation matrices, $e^{\hat{\xi}_i(0)}\forall i$ are positive definite
  - ~~(A2) $|v_i| = 1\ \forall i$ each agent's speed is constant and normalized.~~
  - (A2)Graph is balanced, fixed, strongly connected and each weights are positive. i.e. $w_{ij} > 0\ \ \forall i, j$
  - ~~(A4) Elements of left eigenvector of the following matrix associated with eigenvalue 0 can be positive.~~

lemma1 Consider the weighted graph and each weights are positive, i.e. $w_{ij} > 0$ $\forall i, j$. If the weighted graph is strongly connected, there exists a positive vector γ such that $L_w^T\gamma = 0$, where L_w : is weighted graph laplacian defined by

$$L_w = \{L_{wij}\} := \begin{cases}\sum_{j\in N_i} w_{ij} & \text{if } i=j\\ -w_{ij} & \text{if } j\in N_i\\ 0 & \text{if } j\notin N_i\end{cases}.$$

Passivity

In this case, the following equation is satisfied.

Lemma2 ([4] pp. 42, Lemma 1) Consider the rigid body motion given by (1). Then the following equation hold for the each rigid body motion.

$$\int_0^T \nu_i^T \mu_i dt \geq -\beta \qquad \beta > 0$$

where $\nu_i^T := \begin{bmatrix} v_i^T & \omega_i^T \end{bmatrix}^T$, $\mu_i^T := \begin{bmatrix} (e^{\hat{\xi}_i} p_i)^T & (\mathrm{sk}(e^{\hat{\xi}_i})^\vee)^T \end{bmatrix}^T$

Proof: This lemma can be easily proven by direct calculation of the derivative of the positive definite function V_i .

$$V_i = \frac{1}{2}\|p_i\|^2 + \phi(e^{\hat{\xi}_i}) := \frac{1}{2}\|p_i\|^2 + \frac{1}{2}\mathrm{tr}(I - e^{\hat{\xi}_i})$$

Let us consider ν_i as the input and μ_i As its output. Then, Lemma2 says that the Basic representation for the rigid body motion is *passive* [5] from the input ν_i to the output μ_i

[4]M. Fujita, H. Kawai and M. W. Spong, ''Passivity-based Dynamic Visual Feedback Control for Three Dimensional Target Tracking: Stability and L2-gain Performance Analysis,'' IEEE *Trans. On Control Systems Technology* Vol. 15, No. 1, pp. 40-52 (2007)

[5] H. Khalil, *Nonlinear Systems*, 2002

Theorem1

Theorem 1 Consider the each rigid body motion given by (1). Under the assumptions (A), the control input (2) achieves output synchronization. Namely $\lim_{t\to\infty}|y_i(t) - y_j(t)| = 0 \quad \forall i, j = 1, \cdots, n$

Sketch of Proof

Define the potential function as the following function

$$V := \sum_{i=1}^{n} \gamma_i \left(\frac{1}{2k_{pi}} p_i^T p_i + \frac{1}{k_{ei}} \phi(e^{\hat{\xi}_i}) \right)$$

where $\gamma_i \quad \forall i = \{1, \cdots, n\}$ are elements of a positive vector such that

$$\gamma^T L_w = 0^T \qquad \gamma^T = \begin{bmatrix} \gamma_1 & \cdots & \gamma_n \end{bmatrix} \qquad L_w = \{L_{wij}\} = \begin{cases} \sum_{j \in N_i} w_{ij} & \text{if } i = j \\ -w_{ij} & \text{if } j \in N_i \\ 0 & \text{if } j \notin N_i \end{cases}$$

$$\dot{V} := \sum_{i=1}^{n} \gamma_i \begin{bmatrix} p_i^T & (\mathrm{sk}(e^{\hat{\xi}_i})^\vee)^T \end{bmatrix} \begin{bmatrix} e^{\hat{\xi}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} k_{pi}^{-1} I & 0 \\ 0 & k_{ei}^{-1} I \end{bmatrix} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}$$

Theorem1

$$\dot{V} := \sum_{i=1}^{n} \gamma_i \begin{bmatrix} p_i^T & (\mathrm{sk}(e^{\hat{\xi}_i})^\vee)^T \end{bmatrix} \begin{bmatrix} e^{\hat{\xi}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} k_{pi}^{-1} I & 0 \\ 0 & k_{ei}^{-1} I \end{bmatrix} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}$$

From $\begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = \sum_{j \in N_i} w_{ij} \begin{bmatrix} k_{pi} I & 0 \\ 0 & k_{ei} I \end{bmatrix} \begin{bmatrix} e^{-\hat{\xi}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p_j - p_i \\ \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \end{bmatrix}$

$$= \sum_{i=1}^{n} \gamma_i \begin{bmatrix} p_i^T & (\mathrm{sk}(e^{\hat{\xi}_i})^\vee)^T \end{bmatrix} \begin{bmatrix} e^{\hat{\xi}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} k_{pi}^{-1} I & 0 \\ 0 & k_{ei}^{-1} I \end{bmatrix} \sum_{j \in N_i} w_{ij} \begin{bmatrix} k_{pi} I & 0 \\ 0 & k_{ei} I \end{bmatrix} \begin{bmatrix} e^{-\hat{\xi}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p_j - p_i \\ \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \end{bmatrix}$$

$$= \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \begin{bmatrix} p_i^T & (\mathrm{sk}(e^{\hat{\xi}_i})^\vee)^T \end{bmatrix} \begin{bmatrix} p_j - p_i \\ \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \end{bmatrix}$$

$$= \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \left(\underline{p_i^T (p_j - p_i)} + (\mathrm{sk}(e^{\hat{\xi}_i})^\vee)^T \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right)$$

$$= -\frac{1}{2}\|p_i\|^2 + \frac{1}{2}\|p_j\|^2 - \frac{1}{2}\|(p_i - p_j)\|^2$$

Theorem1

Using $a^T b = -\frac{1}{2}\mathrm{tr}(\hat{a}\hat{b})$, we can show

$$\sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i (\mathrm{sk}(e^{\hat{\xi}_i})^\vee)^T \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) = -\sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \phi(e^{\hat{\xi}_i}) + \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \phi(e^{\hat{\xi}_j})$$
$$- \frac{1}{4} \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \mathrm{tr}\left((e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})(I - e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right)$$

So, the following equation is satisfied.

$$\dot{V} = \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \left(p_i^T (p_j - p_i) + (\mathrm{sk}(e^{\hat{\xi}_i})^\vee)^T \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right)$$

$$= \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \left(\underbrace{-\frac{1}{2}\|p_i\|^2 - \phi(e^{\hat{\xi}_i})}_{= -V_i} + \underbrace{\frac{1}{2}\|p_j\|^2 + \phi(e^{-\hat{\xi}_i})}_{= V_j} - \frac{1}{2}\|(p_i - p_j)\|^2 - \frac{1}{2}\mathrm{tr}\left((e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})(I - e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right) \right)$$

$$= \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \left(-V_i + V_j - \frac{1}{2}\|(p_i - p_j)\|^2 - \frac{1}{2}\mathrm{tr}\left((e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})(I - e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right) \right)$$

Theorem1

Equation

$$\sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i (-V_i + V_j)$$

can be changed to

$$-\gamma^T L_w \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \qquad \gamma^T L_k = 0^T \qquad L_k = \{L_{kij}\} = \begin{cases} \sum_{j \in N_i} k_{ij} & \text{if } i = j \\ -k_{ij} & \text{if } i \in N_j \\ 0 & \text{if } i \notin N_i \end{cases}$$

So

$$\sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i (-V_i + V_j) = 0$$

Consequently

$$\dot{V} = \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \left(-\frac{1}{2}\|(p_i - p_j)\|^2 - \frac{1}{2}\mathrm{tr}\left((e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})(I - e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right) \right)$$

Now rotation matrices $e^{\hat{\xi}_i} \ \forall i$ are assumed to be positive definite, therefore they satisfy the following inequality

$$\lambda_{\min}(B + B^T)\mathrm{tr}(A) \leq \mathrm{tr}((B + B^T)A) \leq \lambda_{\max}((B + B^T))\mathrm{tr}(A)$$

Theorem1

Therefore the derivative of the potential function reduces to

$$\dot{V} \leq \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \left(-\frac{1}{2}\|(p_i - p_j)\|^2 - \frac{1}{2}\lambda_{\min}(e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})\phi(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right)$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \left(\|(p_i - p_j)\|^2 + \lambda_{\min}(e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})\phi(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right) \leq 0$$

Using LaSalle's Invariance Principle

$$0 = \dot{V} \leq \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \left(-\frac{1}{2}\|(p_i - p_j)\|^2 - \frac{1}{2}\lambda_{\min}(e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})\phi(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right) \leq 0$$

$$-\frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} \gamma_i \left(-\frac{1}{2}\|(p_i - p_j)\|^2 - \frac{1}{2}\lambda_{\min}(e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})\phi(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \right) = 0$$

$$\gamma_i > 0$$

$$\dot{V} = 0 \Leftrightarrow \|(p_i - p_j)\|^2 = 0, \phi(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) = 0 \quad (i, j) \in E \qquad \lambda_{\min}(e^{\hat{\xi}_i} + e^{-\hat{\xi}_i}) > 0$$

$$w_{ij} > 0$$

Now the graph is assumed strongly connected, so

$$\begin{matrix} \|(p_i - p_j)\|^2 = 0 \\ \phi(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) = 0 \end{matrix} \quad (i, j) \in E \quad \Longrightarrow \quad \begin{matrix} \|(p_i - p_j)\|^2 = 0 \\ \phi(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) = 0 \end{matrix} \quad \forall i, j$$

Output Synchronization in SE(3) -Passivity Approach-

Theorem1

$\|(p_i - p_j)\|^2 = 0$, $\phi(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) = 0$ means position and orientation of every i-th rigid body converge to the same value and now $y_i = [p_i \quad e^{\hat{\xi}_i}]$.

Each agents converge to the same output

Q.E.D.

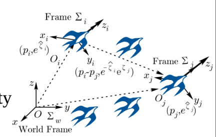
Summary

- **Rigid Body Motion** $(i = 1, \cdots, n)$

$$\dot{p}_i = e^{\hat{\xi}_i} v_i^b$$
$$\dot{e}^{\hat{\xi}_i} = e^{\hat{\xi}_i} \hat{\omega}_i^b \quad (1)$$
$$\varsigma_i = \theta_i \xi_i$$
$$y_i = [p_i \quad e^{\hat{\xi}_i}]$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}^\wedge := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$p_i \in \mathcal{R}^3$ position
$e^{\hat{\xi}_i} \in SO(3)$ orientation
$v_i^b \in \mathcal{R}^3$ body velocity
$\omega_i^b \in \mathcal{R}^3$ angular velocity
$\theta_i \in \mathcal{R}$ rotation angle
$\xi_i \in \mathcal{R}^3$ rotation axes
$y_i \in SE(3)$ output



- **Control Input**

$$\begin{bmatrix} v_i^b \\ \omega_i^b \end{bmatrix} = \sum_{j \in N_i} w_{ij} \begin{bmatrix} k_{pi} I & 0 \\ 0 & k_{ei} I \end{bmatrix} \begin{bmatrix} e^{-\hat{\xi}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p_j - p_i \\ \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j})^\vee \end{bmatrix}$$

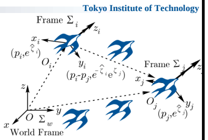
$k_{pi} > 0$
$k_{ei} > 0$
$\forall i = \{1, \cdots, n\}$

- **Assumptions** (A)
  - (A1) At the initial time $t = 0$, the agents' orientation matrices, $e^{\hat{\xi}_i(0)}$ $\forall i$ are positive definite
  - (A2)Graph is fixed, strongly connected and each weights are positive. i.e. $w_{ij} > 0 \quad \forall i, j$
- **Potential Function**

$$V := \sum_{i=1}^{n} \gamma_i \left(\frac{1}{2k_{pi}} p_i^T p_i + \frac{1}{k_{ei}} \phi(e^{\hat{\xi}_i}) \right)$$

- **Goal** $\lim_{t \to \infty} |y_i(t) - y_j(t)| = 0$

Outline



Extension of Output Synchronization in SE(3)

In section 3, we choose the input as

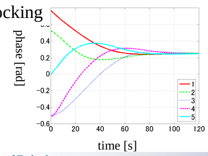
$$\begin{bmatrix} v_i^b \\ \omega_i^b \end{bmatrix} = \sum_{j \in N_i} w_{ij} \begin{bmatrix} k_{pi} I & 0 \\ 0 & k_{ei} I \end{bmatrix} \begin{bmatrix} e^{-\hat{\xi}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p_j - p_i \\ \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j})^\vee \end{bmatrix}.$$

If output synchronization is achieved i.e $\lim_{t \to \infty} |y_i(t) - y_j(t)| = 0 \quad \forall i, j = 1, \cdots, n$, then

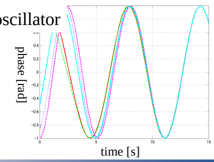
$$\begin{bmatrix} v_i^b \\ \omega_i^b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It means that if output synchronization is achieved, then all output don't change.
In Kuramoto oscillator, phase changes even if synchronization is achieved.

Flocking



Kuramoto oscillator



Refine the control input

We change the input to the following.

$$\begin{bmatrix} v_i^b \\ \omega_i^b \end{bmatrix} = \begin{bmatrix} e^{-\hat{\xi}_i} & 0 \\ 0 & e^{-\hat{\xi}_i} \end{bmatrix} \begin{bmatrix} v_c(t) \\ e^{-\hat{\xi}_c(t)} \omega_c(t) \end{bmatrix} + \sum_{j \in N_i} w_{ij} \begin{bmatrix} k_{pi} I & 0 \\ 0 & k_{ei} I \end{bmatrix} \begin{bmatrix} e^{-\hat{\xi}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p_j - p_i \\ \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \end{bmatrix}$$

$\omega_c(t) := e^{-\hat{\xi}_c(t)} \dot{e}^{\hat{\xi}_c(t)}$

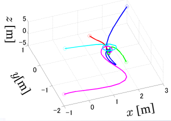
$e^{-\hat{\xi}_c(t)}$, $v_c(t)$ are the same value for all rigid body. (3)

If output synchronization is achieved i.e $\lim_{t \to \infty} |y_i(t) - y_j(t)| = 0 \quad \forall i, j = 1, \cdots, n$, then

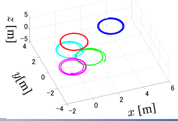
$$\begin{bmatrix} v_i^b \\ \omega_i^b \end{bmatrix} = \begin{bmatrix} e^{-\hat{\xi}_i} & 0 \\ 0 & e^{-\hat{\xi}_i} \end{bmatrix} \begin{bmatrix} v_c(t) \\ e^{-\hat{\xi}_c(t)} \omega_c(t) \end{bmatrix}$$

Using this input, the output change even if output synchronization is achieved.

$v_c = 0$
$\omega_c = 0$



$v_c = 0$
$\omega_c \neq 0$



Refine the control input

$$\begin{bmatrix} v_i^b \\ \omega_i^b \end{bmatrix} = \begin{bmatrix} e^{-\hat{\xi}_i} & 0 \\ 0 & e^{-\hat{\xi}_i} \end{bmatrix} \begin{bmatrix} v_c(t) \\ e^{-\hat{\xi}_c(t)} \omega_c(t) \end{bmatrix} + \sum_{j \in N_i} w_{ij} \begin{bmatrix} k_{pi} I & 0 \\ 0 & k_{ei} I \end{bmatrix} \begin{bmatrix} e^{-\hat{\xi}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p_j - p_i \\ \mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j}) \end{bmatrix}$$

- Attitude Coordination
  - $|v_i| = 1 \; \forall i$ each agent's speed is constant and normalized.
  - $\omega_c = 0$
- Kuramoto Oscillator
  - Not consider about positions.
  - $e^{\hat{\xi}_i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $e^{\hat{\xi}_c} = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $w_{ij} k_{ei} = \frac{K}{N}$

K : Gain
N : The number of oscillator

- Consensus
  - Not consider about orientations.
  - $v_c = 0$

Output Synchronization in SE(3) -Passivity Approach-

Relationship between Flocking and Kuramoto Oscillator

Now we try to prove that output synchronization is achieved using the input (3).
When we prove the convergence, we use the following idea.

Flocking in 2D, the closed loop is (* We omit the grain to be simplified.)

$$\dot\theta_i = -\sum_{j\in N_i}\sin(\theta_i-\theta_j)\,. \qquad (4)$$

Kuramoto Oscillator's model is (* We assume the natural frequencies ω_i are the same for all oscillator.)

$$\dot\theta_i = \omega - \sum_{j\in N_i}\sin(\theta_i-\theta_j) \qquad (5)$$

We can translate from the equation (5) to the equation (4).

Consider $\psi_i := \theta_i - \int_0^t \omega dt.$

$$\dot\psi_i = \dot\theta_i - \omega = -\sum_{j\in N_i}\sin(\theta_i-\theta_j) = -\sum_{j\in N_i}\sin(\theta_i - \int_0^t\omega dt - \theta_j + \int_0^t \omega dt) = -\sum_{j\in N_i}\sin(\psi_i-\psi_j)$$

$$\dot\psi_i = -\sum_{j\in N_i}\sin(\psi_i-\psi_j)$$

⇕

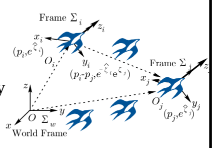
$$\dot\theta_i = -\sum_{j\in N_i}\sin(\theta_i-\theta_j)$$

Translation

- **Rigid Body Motion** $(i=1,\cdots,n)$

$$\dot p_i = e^{\hat\xi_i}v_i^b$$
$$\dot e^{\hat\xi_i} = e^{\hat\xi_i}\hat\omega_i^b \qquad (1)$$
$$\varsigma_i = \theta_i\xi_i$$
$$y_i = \begin{bmatrix} p_i & e^{\hat\xi_i}\end{bmatrix}$$
$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}^\wedge := \begin{bmatrix} 0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0\end{bmatrix}$$

$p_i \in \mathcal{R}^3$ position
$e^{\hat\xi_i}\in SO(3)$ orientation
$v_i^b\in\mathcal{R}^3$ body velocity
$\omega_i^b \in \mathcal{R}^3$ angular velocity
$\theta_i\in\mathcal{R}$ rotation angle
$\xi_i \in \mathcal{R}^3$ rotation axes
$y_i \in SE(3)$ output



- **Control Input**

$$\begin{bmatrix} v_i^b \\ \omega_i^b\end{bmatrix} = \begin{bmatrix} e^{-\hat\xi_i} & 0\\ 0 & e^{-\hat\xi_i}\end{bmatrix}\begin{bmatrix} v_c(t) \\ e^{-\hat\xi_c(t)}\omega_c(t)\end{bmatrix} + \sum_{j\in N_i} w_{ij}\begin{bmatrix} k_{pi}I & 0\\ 0 & k_{ei}I\end{bmatrix}\begin{bmatrix} e^{-\hat\xi_i} & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} p_j - p_i \\ \mathrm{sk}(e^{-\hat\xi_i}e^{\hat\xi_j})\end{bmatrix}$$

- **Closed Loop (position)**

$$\dot p_i = v_c(t) + \sum_{j\in N_i} w_{ij}k_{pi}(p_j - p_i)$$

Translation

Now we consider $\bar p_i := p_i - \int_0^t v_c dt$.

$$v_i = e^{-\hat\xi_i}v_c(t) + \sum_{j\in N_i} w_{ij}k_{pi}e^{-\hat\xi_i}(p_j-p_i)$$

$$\frac{d}{dt}\left(p_i - \int_0^t v_c dt\right) = \dot p_i - v_c$$
$$= e^{\hat\xi_i}v_i - v_c$$
$$= e^{\hat\xi_i}\left(e^{-\hat\xi_i}v_c(t) + \sum_{j\in N_i}w_{ij}k_{pi}e^{-\hat\xi_i}(p_j-p_i)\right) - v_c$$
$$= v_c(t) + \sum_{j\in N_i}w_{ij}k_{pi}(p_j-p_i) - v_c$$
$$= \sum_{j\in N_i}w_{ij}k_{pi}(p_j-p_i)$$
$$= \sum_{j\in N_i}w_{ij}k_{pi}\left(p_j - \int_0^t v_c dt - \left(p_i - \int_0^t v_c dt\right)\right)$$

$$\frac{d}{dt}\left(p_i - \int_0^t v_c dt\right) = \sum_{j\in N_i}w_{ij}k_{pi}\left(p_j - \int_0^t v_c dt - \left(p_i - \int_0^t v_c dt\right)\right)$$

Translation

$$\frac{d}{dt}\left(p_i - \int_0^t v_c dt\right) = \sum_{j\in N_i}w_{ij}k_{pi}\left(p_j - \int_0^t v_c dt - \left(p_i - \int_0^t v_c dt\right)\right) \qquad \bar p_i := p_i - \int_0^t v_c dt$$

$$\dot{\bar p}_i = \sum_{j\in N_i}w_{ij}k_{pi}(\bar p_i - \bar p_j)$$

⇕

$$\dot p_i = \sum_{j\in N_i}w_{ij}k_{pi}(p_j - p_i) \qquad (\text{if } v_c = 0)$$

Next we consider $\bar e^{\hat\xi_i} := e^{-\hat\xi_c}e^{\hat\xi_i}$

$$\frac{d}{dt}\left(e^{-\hat\xi_c}e^{\hat\xi_i}\right) = \dot e^{-\hat\xi_c}e^{\hat\xi_i} + e^{-\hat\xi_c}\dot e^{\hat\xi_i}$$
$$= \dot e^{-\hat\xi_c}e^{\hat\xi_i} + e^{-\hat\xi_c}\dot e^{\hat\xi_i}$$
$$= (e^{\hat\xi_c}\hat\omega_c)^T e^{\hat\xi_i} + e^{-\hat\xi_c}(e^{\hat\xi_i}\hat\omega_i)$$
$$= -\hat\omega_c e^{-\hat\xi_c}e^{\hat\xi_i} + e^{-\hat\xi_c}e^{\hat\xi_i}\hat\omega_i$$

Translation

$$= -\hat\omega_c e^{-\hat\xi_c}e^{\hat\xi_i} + e^{-\hat\xi_c}e^{\hat\xi_i}\hat\omega_i$$
$$= -\hat\omega_c e^{-\hat\xi_c}e^{\hat\xi_i} + e^{-\hat\xi_c}e^{\hat\xi_i}\left\{(e^{-\hat\xi_c}e^{\hat\xi_i})^T\hat\omega_c(e^{-\hat\xi_c}e^{\hat\xi_i}) + \hat u\right\}$$
$$= -\hat\omega_c e^{-\hat\xi_c}e^{\hat\xi_i} + \hat\omega_c(e^{-\hat\xi_c}e^{\hat\xi_i}) + e^{-\hat\xi_c}e^{\hat\xi_i}\hat u$$
$$= e^{-\hat\xi_c}e^{\hat\xi_i}\hat u$$

$$\omega_i = e^{-\hat\xi_i}e^{\hat\xi_c}\omega_c + u_i$$
$$= (e^{-\hat\xi_c}e^{\hat\xi_i})^T\omega_c + u_i$$
$$u_i = \sum_{j\in N_i}w_{ij}k_{ei}\left(\mathrm{sk}(e^{-\hat\xi_i}e^{\hat\xi_j})\right)$$
$$\hat\omega_i = (e^{-\hat\xi_c}e^{\hat\xi_i})^T\hat\omega_c(e^{-\hat\xi_c}e^{\hat\xi_i}) + \hat u$$

$$\frac{d}{dt}\left(e^{-\hat\xi_c}e^{\hat\xi_i}\right) = e^{-\hat\xi_c}e^{\hat\xi_i}\hat u \qquad u_i = \sum_{j\in N_i}w_{ij}k_{ei}\left(\mathrm{sk}\left((e^{-\hat\xi_c}e^{\hat\xi_i})^T(e^{-\hat\xi_c}e^{\hat\xi_j})\right)\right)$$

$$\frac{d}{dt}\left(e^{-\hat\xi_c}e^{\hat\xi_i}\right) = e^{-\hat\xi_c}e^{\hat\xi_i}\sum_{j\in N_i}w_{ij}k_{ei}\left(\mathrm{sk}\left((e^{-\hat\xi_c}e^{\hat\xi_i})^T(e^{-\hat\xi_c}e^{\hat\xi_j})\right)\right)^\wedge \qquad \bar e^{\hat\xi_i} := e^{-\hat\xi_c}e^{\hat\xi_i}$$

$$\dot{\bar e}^{\hat\xi_i} = \bar e^{\hat\xi_i}\sum_{j\in N_i}w_{ij}k_{ei}\left(\mathrm{sk}\left((\bar e^{\hat\xi_i})^T(\bar e^{\hat\xi_j})\right)\right)^\wedge$$

⇕

$$\dot e^{\hat\xi_i} = e^{\hat\xi_i}\sum_{j\in N_i}w_{ij}k_{ei}\left(\mathrm{sk}\left((e^{\hat\xi_i})^T(e^{\hat\xi_j})\right)\right)^\wedge$$

Translation

Consequently if we use the control input (3), then the closed loop system is

$$\begin{cases}\dot{\bar p}_i = \sum_{j\in N_i}w_{ij}k_{pi}(\bar p_i - \bar p_j)\\ \dot{\bar e}^{\hat\xi_i} = \bar e^{\hat\xi_i}\sum_{j\in N_i}w_{ij}k_{ei}\left(\mathrm{sk}\left((\bar e^{\hat\xi_i})^T(\bar e^{\hat\xi_j})\right)\right)^\wedge\end{cases} \Leftrightarrow \begin{cases}\dot p_i = \sum_{j\in N_i}w_{ij}k_{pi}(p_i - p_j)\\ \dot e^{\hat\xi_i} = e^{\hat\xi_i}\sum_{j\in N_i}w_{ij}k_{ei}\left(\mathrm{sk}\left((e^{\hat\xi_i})^T(e^{\hat\xi_j})\right)\right)^\wedge\end{cases}$$

So we can show

$$\lim_{t\to\infty}(\bar p_i - \bar p_j) = 0 \quad \lim_{t\to\infty}(\bar e^{\hat\xi_i} - \bar e^{\hat\xi_j}) = 0 \qquad \bar V := \sum_{i=1}^n \gamma_i\left(\frac{1}{2k_{pi}}\bar p_i^T\bar p_i + \frac{1}{k_{ei}}\phi(\bar e^{\hat\xi_i})\right)$$

using the same way of proof in theorem1.

$$\begin{matrix}\lim_{t\to\infty}(\bar p_i - \bar p_j) = 0 \\ \lim_{t\to\infty}(\bar e^{\hat\xi_i} - \bar e^{\hat\xi_j}) = 0\end{matrix} \Leftrightarrow \begin{matrix}\lim_{t\to\infty}(p_i - p_j) = 0 \\ \lim_{t\to\infty}(e^{\hat\xi_i} - e^{\hat\xi_j}) = 0\end{matrix}$$

$\left\|(p_i - p_j)\right\|^2 = 0$, $\phi(e^{-\hat\xi_i}e^{\hat\xi_j}) = 0$ means position and orientation of every *i*-th rigid body converge to the same value and now $y_i = [p_i \quad e^{\hat\xi_i}]$.

Each agents converge to the same output

Output Synchronization in SE(3) -Passivity Approach-

Theorem2

Consequently the following theorem is satisfied.

Theorem 1 Consider the each rigid body motion given by (1). Under the assumptions (A), the control input (3) achieves output synchronization. Namely $\lim_{t\to\infty}|y_i(t)-y_j(t)|=0 \quad \forall i,j=1,\cdots,n$

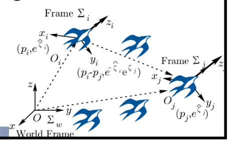
Sketch of Proof

Define the potential function as the following function

$$\bar{V} := \sum_{i=1}^{n}\gamma_i\left(\frac{1}{2k_{pi}}\bar{p}_i^T\bar{p}_i+\frac{1}{k_{ei}}\phi(\bar{e}^{\hat{\xi}_i})\right)$$

After that, we can prove using the same way of theorem1.

Outline



Simulation

Simulation

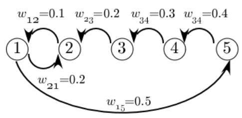
$p_1(0)=\begin{bmatrix}1\\0\\3\end{bmatrix}$ $\zeta_1(0)=\begin{bmatrix}-0.21\\-0.50\\0.77\end{bmatrix}$

$p_2(0)=\begin{bmatrix}2\\-1\\2\end{bmatrix}$ $\zeta_2(0)=\begin{bmatrix}0.52\\-0.76\\0.52\end{bmatrix}$

$p_3(0)=\begin{bmatrix}3\\1\\2\end{bmatrix}$ $\zeta_3(0)=\begin{bmatrix}-0.21\\0.77\\-0.50\end{bmatrix}$

$p_4(0)=\begin{bmatrix}-1\\-2\\0\end{bmatrix}$ $\zeta_4(0)=\begin{bmatrix}-0.14\\0.51\\-0.51\end{bmatrix}$

$p_5(0)=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\zeta_5(0)=\begin{bmatrix}0\\1.05\\0\end{bmatrix}$

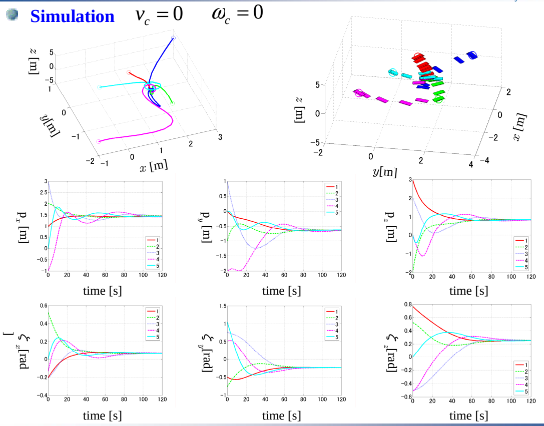


$$L_w=\begin{bmatrix}0.1 & -0.1 & 0 & 0 & 0\\ -0.2 & 0.4 & -0.2 & 0 & 0\\ 0 & 0 & 0.3 & -0.3 & 0\\ 0 & 0 & 0 & 0.4 & -0.4\\ -0.5 & 0 & 0 & 0 & 0.5\end{bmatrix}$$

$$\gamma=\begin{bmatrix}0.9466\\0.2367\\0.1578\\0.1183\\0.0947\end{bmatrix} \quad k_{pi}=0.5 \quad k_{ei}=0.5$$

Simulation

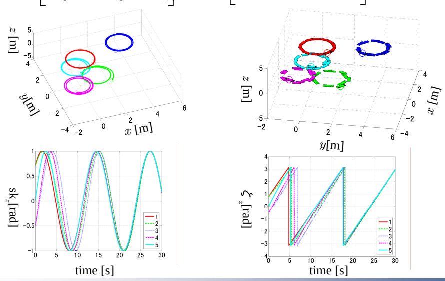
Simulation $v_c=0 \quad \omega_c=0$



Simulation

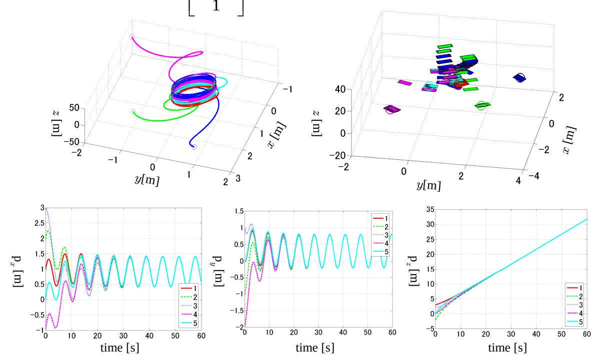
Simulation

- Kuramoto Oscillator
  - $|v_i|=1\ \forall i$ each agent's speed is constant and normalized.
  - $e^{\hat{\xi}_i}=\begin{bmatrix}\cos\theta_i & -\sin\theta_i & 0\\ \sin\theta_i & \cos\theta_i & 0\\ 0 & 0 & 1\end{bmatrix}, \quad e^{\hat{\xi}_c}=\begin{bmatrix}\cos\omega t & -\sin\omega t & 0\\ \sin\omega t & \cos\omega t & 0\\ 0 & 0 & 1\end{bmatrix}$



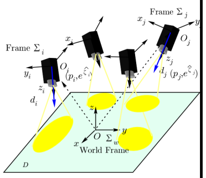
Simulation

Simulation $v_c=\begin{bmatrix}\cos\omega t\\ \sin\omega t\\ 1\end{bmatrix} \quad \omega_c=0$

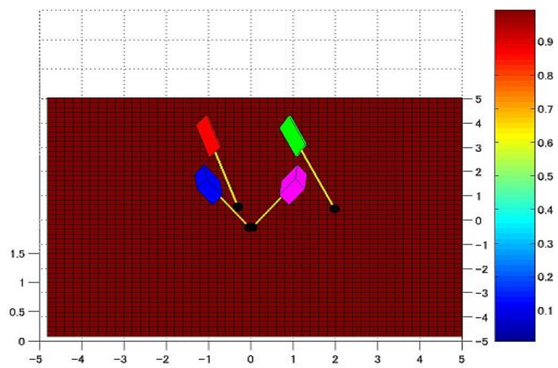


Output Synchronization in SE(3) -Passivity Approach-

Outline



Simulation



Problem Statement

- **Rigid Body Motion in visual sensor** $(i = 1, \cdots, n)$

$$\dot{p}_i = 0, \quad \dot{e}^{\hat{\xi}_i} = e^{\hat{\xi}_i}\hat{\omega}_i^b, \quad \varsigma_i = \theta_i \xi_i \qquad (1)$$

$p_i \in \mathcal{R}^3$ position

$e^{\hat{\xi}_i} \in SO(3)$ orientation

$\omega_i^b \in \mathcal{R}^3$ angular velocity

$\theta_i \in \mathcal{R}$ rotation angle

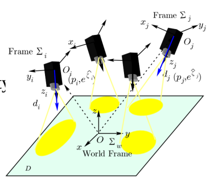
$\xi_i \in \mathcal{R}^3$ rotation axes

$d_i \in \mathcal{R}^3$ light of sight

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}^{\wedge} := \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}$$

$\mathcal{D}$: compact subset of $\mathcal{R}^2$ which represents a region in $\mathcal{R}^2$ that the visual sensors are required to cover.

q: points of region of $\mathcal{D}$

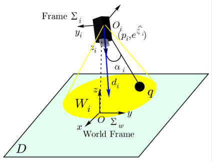


- **View Angle**

View Angle of the visual sensor is represented by

$$\alpha_i := \cos^{-1}\frac{(q-p_i)^T e^{\hat{\xi}_i} d_i}{\|q-p_i\|}$$

$$s_i := \frac{(q-p_i)^T e^{\hat{\xi}_i} d_i}{\|q-p_i\|} = \cos\alpha_i$$



Sensor Model

- **Sensor Model** $A_i(s_i)$ $(i = 1, \cdots, n)$

SM1: Each visual sensor has a peak sensing capacity of M_i exactly at the view angle s_i of the visual sensor. That is, we have

$$A_i(s_i) = M_i > A(s_i) \qquad \forall s_i < 1$$

SM2: Each visual sensor has a *limited sensory domain* $W_i(t)$ with a limited visual angle γ_i. The sensory domain of each visual sensor is given by

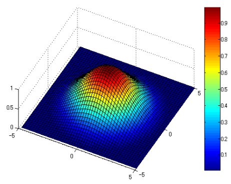
$$W_i(t) = \{q \in D : \alpha_i \le \gamma_i\}$$

SM3: Sensor Model $A_i(s_i)$ is positive $\forall q \in W_i$ and 0 $\forall q \notin W_i$.

Example

$$A_i(s_i) = \begin{cases}\dfrac{M_i}{(1-r_i)^2}(s_i - r_i)^2 & \text{if } s \ge r_i\\ 0 & \text{if } s < r_i\end{cases}$$

$$r_i = \cos\gamma_i$$



Information Model

- **Information Model**

The effective coverage achieved by a visual sensor surveying q from the initial time $t_0 = 0$ to time t is defined to be

$$I_i(q,t) := \int_0^t A_i(s_i)dt \qquad I_i(q,t): \text{ Information map}$$

and the effective coverage by a sub set of visual sensor N_i in surveying q is then given by

$$I_{N_i}(q,t) = \sum_{j \in N_i} I_j(q,t) = \int_0^t \sum_{j \in N_i} A_i(s_i)dt$$

$$\frac{\partial}{\partial t} I_{N_i}(q,t) = \sum_{j \in N_i} A_i(s_i) \ge 0 \qquad A_i(s_i): \text{ Sensor map of a visual sensor.}$$

We assume

IC1: $I_{N_i}(q,t) = 0 \quad q \in D$

Objective Function

- **Objective Function (error function)**

$$e(t) := \int_D h\left(C^* - I_S(q,t)\right)\phi(q)dq \qquad S := \{i \mid i = \{1, \cdots, n\}\}$$

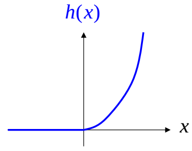
where $h(x)$ have the following properties.

- $h(x) > 0, \ \dfrac{\partial}{\partial x}h(x) > 0, \ \dfrac{\partial^2}{\partial^2 x}h(x) > 0$
- $h(x) = \dfrac{\partial}{\partial x}h(x) = \dfrac{\partial^2}{\partial^2 x}h(x) = 0 \qquad \forall x \le 0$

Example

$$h(x) = \left(\max(0,x)^\beta\right)^\beta \qquad \beta > 1$$

$$h(x) = \begin{cases} e^x - 1 & \text{if } x \ge 0\\ 0 & \text{if } x < 0\end{cases}$$



- **Goal**

$$\lim_{t \to \infty} e(t) = 0$$

Output Synchronization in SE(3) -Passivity Approach-

Control Input

- **Control Input** (Fully connected case)

$$\omega_i = K_i \int_D \frac{\partial}{\partial x} h(x)\Big|_{x=C^*-I_s(q,t)} \frac{\partial}{\partial s_i} A_i(s_i) \frac{\hat{d}_i^T e^{-\hat{\xi}_i}(q-p_i)}{\|q-p_i\|}\phi(q)dq \qquad K_i > 0 \qquad (2)$$

- Integral evaluation over $W_i(t)$ instead over D sufficient.

$$\left(\frac{\partial}{\partial s_i} A_i(s_i) \neq 0 \text{ only in } W_i(t) \right)$$

- Input is zero if $I_S(q,t) = C^*$ in $W_i(t)$.
- $\phi(q): D \to R^+$ Distribution density function.

Theorem

Theorem 1 Consider the each rigid body motion in a visual sensor given by (1). Under the Sensor Model SM1-3 and IC1, the control input (2) achieves $I_S(q,t) = C^*$ in $W_i(t)$.

Sketch of Proof

Define the potential function as the following function

$$V = -\dot{e}(t)$$
$$= -\frac{d}{dt}\int_D h\left(C^* - I_S(q,t)\right)\phi(q)dq$$
$$= \sum_{i\in S}\int_D \frac{\partial}{\partial x} h(x)\Big|_{x=C^*-I_s(q,t)} A_i(s_i)\phi(q)dq \geq 0$$

$$\dot{V} = -\ddot{e}(t)$$
$$= -\int_D \frac{\partial^2}{\partial^2 x} h(x)\Big|_{x=C^*-I_s(q,t)} \left(\sum_{i\in S} A_i(s_i)\right)^2 \phi(q)dq$$
$$-\sum_{i\in S}\left[\left(\int_D \frac{\partial}{\partial x} h(x)\Big|_{x=C^*-I_s(q,t)} \frac{\partial}{\partial s_i} A_i(s_i) \frac{(q-p_i)^T e^{\hat{\xi}_i}\hat{d}_i}{\|q-p_i\|}\phi(q)dq\right)\omega_i\right]$$

Theorem

$$\dot{V} = -\int_D \frac{\partial^2}{\partial^2 x} h(x)\Big|_{x=C^*-I_s(q,t)} \left(\sum_{i\in S} A_i(s_i)\right)^2 \phi(q)dq$$
$$-\sum_{i\in S}\left[\left(\int_D \frac{\partial}{\partial x} h(x)\Big|_{x=C^*-I_s(q,t)} \frac{\partial}{\partial s_i} A_i(s_i) \frac{(q-p_i)^T e^{\hat{\xi}_i}\hat{d}_i}{\|q-p_i\|}\phi(q)dq\right)\omega_i\right]$$

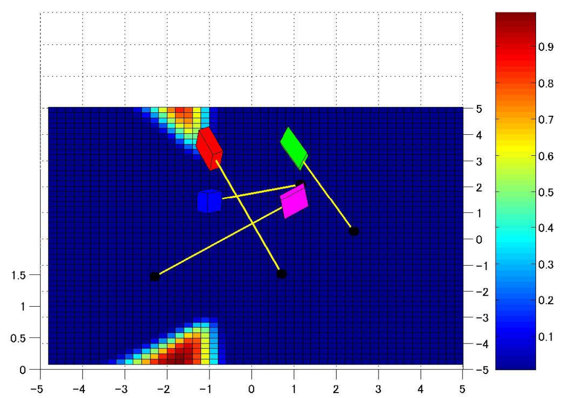
$$\omega_i = \int_D \frac{\partial}{\partial x} h(x)\Big|_{x=C^*-I_s(q,t)} \frac{\partial}{\partial s_i} A_i(s_i) \frac{\hat{d}_i^T e^{-\hat{\xi}_i}(q-p_i)}{\|q-p_i\|}\phi(q)dq$$

$$\dot{V} = -\int_D \frac{\partial^2}{\partial^2 x} h(x)\Big|_{x=C^*-I_s(q,t)} \left(\sum_{i\in S} A_i(s_i)\right)^2 \phi(q)dq$$
$$-\sum_{i\in S}\left[\left(\int_D \frac{\partial}{\partial x} h(x)\Big|_{x=C^*-I_s(q,t)} \frac{\partial}{\partial s_i} A_i(s_i) \frac{(q-p_i)^T e^{\hat{\xi}_i}\hat{d}_i}{\|q-p_i\|}\phi(q)dq\right)^T\right.$$
$$\left.\left(\int_D \frac{\partial}{\partial x} h(x)\Big|_{x=C^*-I_s(q,t)} \frac{\partial}{\partial s_i} A_i(s_i) \frac{(q-p_i)^T e^{\hat{\xi}_i}\hat{d}_i}{\|q-p_i\|}\phi(q)dq\right)\right] \leq 0$$

Theorem

$$\dot{V} = 0 \Leftrightarrow I_S(q,t) = C^* \qquad \forall q \in W_i \qquad \forall i \in S \qquad S := \{i \mid i = \{1,\cdots,n\}\}$$

Simulation



Future Works

1. Experiments
2. Extension to Visual Attitude Coordination
3. Connection this result and game theory or MPC
4. Extension of Effective Coverage Control using Visual Sensor

⋮

and so on.