A condition for collision avoidance and a partial solution to reduction of on-line computation effort

（衝突回避のための条件とオンライン計算量低減化の部分的解法）

11th FL seminar
Takeshi Hatanaka
25 June, 2007

Outline of This talk

- Collision Avoidance Problem
- Problem Formulation
- Definition and Computations of Safe Region
- Problem and Its Partial Solution
- Conclusions and Future Works

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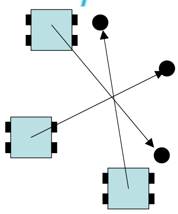
Cooperative Control

Cooperative Control is a control of a set of dynamically decoupled subsystems that are required to perform a cooperative task.



Formation Control:
Its objective is that a group of vehicles or aircrafts cooperatively converges to a desired formation.

Characteristics:

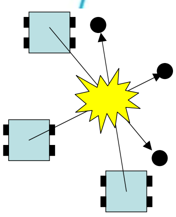
- there are large number of subsystems independently actuated
- subsystems are dynamically decoupled
- objectives can only be achieved through a collective behavior
- feasible set of states depends on the other subsystem's state

One control approach that accommodates a general cooperative objective is receding horizon control (RHC).

Collision Avoidance (CA)

Dynamics of each agent is given by

$$x^i(t+1) = f^i(x^i(t), u^i(t)),\ i \in \{1, \cdots, M\}$$
$$x^i(t) \in \mathcal{X}^i \text{ and } u^i(t) \in \mathcal{U}^i \quad \forall t \in \mathbb{Z}_+$$



Interconnection graph: $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{A}(t)\}$

$\mathcal{V}$: Set of nodes

$\mathcal{A}(t) \subseteq \mathcal{V} \times \mathcal{V}$: Set of time-varying arcs (i, j) at time $t \in \mathbb{Z}_+$

CA Problem via RHC

At each time $t \in \mathbb{Z}_+$, compute a predicted input sequence $u^i_{0,t}, \cdots, u^i_{N-1,0}$ and input $u^i_{0,t}$ to each agent $i \in \{1, 2, \cdots, M\}$

$x^i_{k,t},\ u^i_{k,t}$: k steps prediction state and input of i at time $t \in \mathbb{Z}_+$

$\tilde{x}^i_{k,t},\ \tilde{u}^i_{k,t}$: Prediction vectors associated with the neighboring systems assuming a constant interconnection

Distributed RHC

Borreri et. al. (2006)

$$\tilde{U}^i_t := [u^i_{0,t} \quad \tilde{u}^i_{0,t} \quad \cdots \quad u^i_{N-1,t} \quad \tilde{u}^i_{N-1,t}]$$

$$\min_{\tilde{U}^i_t} \sum_{k=0}^{N-1} l^i_t(x^i_{k,t}, u^i_{k,t}, \tilde{x}^i_{k,t}, \tilde{u}^i_{k,t})$$

$x^i_{k+1,t} = f^i(x^i_{k,t}, u^i_{k,t})$ ・・・Future evolution of agent i

$x^i_{k,t} \in \mathcal{X}^i,\ u^i_{k,t} \in \mathcal{U}^i\ \forall k \in \{1, \cdots, N-1\}$ ・・・Constraint Fulfillment of agent i

$x^j_{k+1,t} = f^j(x^j_{k,t}, u^j_{k,t}),\ (i,j) \in \mathcal{A}(t)$ ・・・Future evolution of neighbors of i

$x^j_{k,t} \in \mathcal{X}^j,\ u^j_{k,t} \in \mathcal{U}^j\ \forall k \in \{1, \cdots, N-1\},\ (i,j) \in \mathcal{A}(t)$ ・・・Constraint fulfillment of neighbors

$g^{i,j}(x^i_{k,t}, u^i_{k,t}, x^j_{k,t}, u^j_{k,t}) \le 0,\ (i,j) \in \mathcal{A}(t), \forall k \in \{1, \cdots, N-1\}$

・・・CA of agent i

$g^{q,r}(x^q_{k,t}, u^q_{k,t}, x^r_{k,t}, u^r_{k,t}) \le 0,\ (q,r) \in \mathcal{A}(t),\ (i,q) \in \mathcal{A}(t), (i,r) \in \mathcal{A}(t)\ \forall k \in \{1, \cdots, N-1\}$

・・・CA of neighbors

$x^i_{N,t} \in \mathcal{X}^i_f,\ x^j_{N,t} \in \mathcal{X}^j_f,\ (i,j) \in \mathcal{A}(t)$ ・・・Terminal Constraint

$x^i_{0,t} = x^i_t,\ \tilde{x}^i_{0,t} = \tilde{x}^i_t$ ・・・Initial Constraint

Previous Works

The neighbors does not always behave as expected

Keviczky et.al. (2006) introduced an additional rule based on the Mixed Integer Program (MIP)

Drawbacks
- ■ No theoretical guarantees
- ■ Large computational effort
- ■ Not optimal

} We try to overcome

Borreri et.al. (2006) presented a switching strategy by using an invariant sets. An emergency controller is employed to achieve CA

⇒ We need to divide the region in advance

We address the CA problem by using a Reference Governor (RG)

⇒ Low complexity＋Theoretical guarantees (if possible)

In this talk, we derive a condition for achieving the CA

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- **Problem Formulation**
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Dynamics of each agent

For simplicity, we assume that all the agent have the same dynamics

$$x^i(t+1) = Ax^i(t) + Br^i(t)$$

$x^i(t) := \begin{bmatrix} x_p^i(t) \\ x_v^i(t) \end{bmatrix}$: State of agent i $\quad i \in \{1, \cdots, M\}$

$r^i(t) \in \mathcal{R} \subset \mathbf{R}^{n_y}$: Reference of position of the i-th agent $\quad n_y \in \{1,2,3\}$

Assumptions (i) A is stable

(ii) $[I \quad O](I-A)^{-1}B = I$ (Integral type servo system)

$$x(t) := \begin{bmatrix} x^1(t) \\ \vdots \\ x^M(t) \end{bmatrix}, r(t) := \begin{bmatrix} r^1(t) \\ \vdots \\ r^M(t) \end{bmatrix}, x_p(t) := \begin{bmatrix} x_p^1(t) \\ \vdots \\ x_p^M(t) \end{bmatrix} \quad \tilde{A} := \begin{bmatrix} A & & O \\ & \ddots & \\ O & & A \end{bmatrix}, \tilde{B} := \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}$$

⇒ $x(t+1) = \tilde{A}x(t) + \tilde{B}r(t)$

$x(t; x(0), r)$: Future evolution of r for the initial state $x(0)$ and step reference x

Constraints

- ■ $x^i(t) \in \mathcal{X}_i \quad \forall i \in \{1,2,3\}, \forall t \in \mathbb{Z}_+$ ・・・General constraints
- ■ $\|x_p^i(t) - x_p^j(t)\|_\infty > 1, \; i \neq j \quad \forall t \in \mathbb{Z}_+$ ・・・CA

$$x(t) \in \mathbb{X} := \{x | \; x^i \in \mathcal{X} \; \forall i\}$$

$$x(t) \in \mathbb{Y} := \{x | \; \|x_p^i - x_p^j\|_\infty > 1 \; \forall i,j\}$$

What kinds of initial states and step references satisfy these constraints?

Maximal Output Admissible Set

$\mathcal{S}_X := \{(x(0), r) | \; r \in \mathbb{R} \text{ and } x(t; x(0), r) \in \mathbb{X} \; \forall t \in \mathbb{Z}_+\}$

$r \in \mathbb{R} := \{r | \; (I-A)^{-1}Br/(1-\epsilon) \in \mathbb{X} \text{ and } \|r^i - r^j\|_\infty > 1 + \epsilon\}, \; \epsilon > 0$

⇒ $\mathcal{S}_X$ can be computed efficiently

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Definition of Safe Region

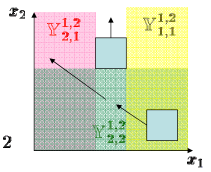
Definition. : Safe Region (SR)

$$\mathcal{S} := \left\{ \begin{bmatrix} x(0) \\ r \end{bmatrix} \middle| \; (x(0), r) \in \mathcal{S}_X \text{ and } x(t; x(0), r) \in \mathbb{Y} \; \forall t \in \mathbb{Z}_+ \right\}$$

Remark : $\mathbb{Y}$ is a non-convex set

$$\mathbb{Y} = \bigcap_{i \neq j \in \{1, \cdots, N\}} \; \bigcup_{l \in \{1,2\}, h \in \{1, \cdots, n_y\}} \mathbb{Y}_{l,h}^{i,j}$$

$$\mathbb{Y}_{l,h}^{i,j} := \begin{cases} \{x | \; x_p^i(h,:) - x_p^j(h,:) > 1\} & \text{if } l = 1 \\ \{x | \; x_p^i(h,:) - x_p^j(h,:) < -1\} & \text{if } l = 2 \end{cases}$$



We have to consider all the pairs of regions as the state evolves

$\mathbb{Y}_{1,1}^{1,2} \longrightarrow \mathbb{Y}_{2,2}^{1,2} \longrightarrow \mathbb{Y}_{2,1}^{1,2}$

$$\mathcal{S}^{i,j} := \left\{ \begin{bmatrix} x(0) \\ r \end{bmatrix} \middle| \; (x(0), r) \in \mathcal{S}_X, \; x(t; x(0), r) \in \mathbb{Y}^{i,j} \; \forall t \in \mathbb{Z}_+ \right\} \quad \mathbb{Y}^{i,j} := \cup_{l,h} \mathbb{Y}_{l,h}^{i,j}$$

⇒ $\mathcal{S} = \bigcap_{i \neq j} \mathcal{S}^{i,j}$

Computations of SR

$$\mathcal{O}_0^{i,j} := \left\{ \begin{bmatrix} x \\ r \end{bmatrix} \middle| \ (x,r) \in \mathcal{S}_X, x \in \mathbb{Y}^{i,j} \right\}$$

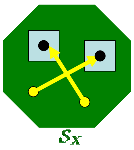
$$\mathcal{O}_{k+1}^{i,j} := \left\{ \begin{bmatrix} x \\ r \end{bmatrix} \middle| \ (x,r) \in \mathcal{S}_X, x \in \mathbb{Y}^{i,j} \text{ and } Ax + Br \in \mathcal{O}_k^{i,j} \right\}$$

$$\mathcal{O}_\infty^{i,j} := \bigcap_{k=0}^{\infty} \mathcal{O}_k^{i,j} \Rightarrow \mathcal{S}^{i,j} = \mathcal{O}_\infty^{i,j}$$

Proposition: $\mathcal{O}_{k+1}^{i,j} = \mathcal{O}_k^{i,j} \Rightarrow \mathcal{O}_\infty^{i,j} = \mathcal{O}_k^{i,j}$

Proposition: $\exists k \in \mathbb{Z}_+ \text{ s.t. } \mathcal{O}_{k+1}^{i,j} = \mathcal{O}_k^{i,j}$

Proof: Immediate from the assumption of
$\mathbb{R} \subseteq \{r \mid \|r^i - r^j\|_\infty > 1 + \epsilon\}, \ \epsilon > 0,$
a property of servo systems and
the definition of convergence



Problem: It is difficult to check $\mathcal{O}_{k+1}^{i,j} = \mathcal{O}_k^{i,j}$ due to their non-convexity

Collision Problem

We consider a collision problem instead of CA

$$\mathcal{C}^{i,j} := \left\{ \begin{bmatrix} x(0) \\ r \end{bmatrix} \middle| (x(0),t) \in \mathcal{S}_X \text{ and } x(t; x(0), r) \in \bar{\mathbb{Y}}^{i,j} \ \forall t \in \mathbb{Z}_+ \right\}$$

$\bar{\mathbb{Y}}^{i,j} := \{x_p \mid \|x_p^i - x_p^j\|_\infty \leq 1\}$: Convex !

Remark $\mathcal{S}^{i,j} = \mathcal{S}_X \setminus \mathcal{C}^{i,j}$

$$\bar{\mathcal{O}}_0^{i,j} := \left\{ \begin{bmatrix} x \\ r \end{bmatrix} \middle| \ (x,r) \in \mathcal{S}_X \text{ and } x \in \bar{\mathbb{Y}}^{i,j} \right\}$$

$$\bar{\mathcal{O}}_{k+1}^{i,j} := \left\{ \begin{bmatrix} x \\ r \end{bmatrix} \middle| \ (x,r) \in \mathcal{S}_X \text{ and } Ax + Br \in \bar{\mathcal{O}}_k^{i,j} \right\}$$

$$\bar{\mathcal{O}}_\infty^{i,j} := \bigcup_{k=0}^{\infty} \bar{\mathcal{O}}_k^{i,j} \Rightarrow \mathcal{C}^{i,j} = \bar{\mathcal{O}}_\infty^{i,j}$$

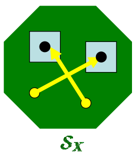
Main Result

Theorem 1: $\bar{\mathcal{O}}_{k+1}^{i,j} \subseteq \bar{\mathcal{O}}_k^{i,j} \Rightarrow \bar{\mathcal{O}}_\infty^{i,j} = \bigcup_{p \in \{0,\cdots,k\}} \bar{\mathcal{O}}_p^{i,j}$

Proof is shown in the next slide

Theorem 2: $\exists k \in \mathbb{Z}_+ \text{ s.t. } \bar{\mathcal{O}}_{k+1}^{i,j} \subseteq \bar{\mathcal{O}}_k^{i,j}$

Proof: It can be proven in the same way as the case of $\mathcal{O}_k^{i,j}$
(A collision can happen in finite time interval)



Point: It is easy to check $\bar{\mathcal{O}}_{k+1}^{i,j} \subseteq \bar{\mathcal{O}}_k^{i,j}$ immediately, because they are convex polyhedra

Remark: $\mathcal{C}^{i,j} = \bar{\mathcal{O}}_\infty^{i,j}$ is not convex (union of convex polyhedra)

Proof of Theorem

$$\bar{\mathcal{O}}_{k+1}^{i,j} \subseteq \bar{\mathcal{O}}_k^{i,j} \Rightarrow \bigcup_{p \in \{0,\cdots,k+1\}} \bar{\mathcal{O}}_p^{i,j} = \bigcup_{p \in \{0,\cdots,k\}} \bar{\mathcal{O}}_p^{i,j} \Leftrightarrow \bar{\mathcal{O}}_\infty^{i,j} = \bigcup_{p \in \{0,\cdots,k\}} \bar{\mathcal{O}}_p^{i,j}$$

Proof: We prove only $(\Rightarrow)$ (The converse is obvious)

We prove $\bar{\mathcal{O}}_\infty^{i,j} \subseteq \bigcup_{p \in \{0,\cdots,k\}} \bar{\mathcal{O}}_p^{i,j}$ (The converse is obvious)

Suppose that there exists $(x,r) \in \bar{\mathcal{O}}_\infty^{i,j} \setminus \bigcup_{p \in \{0,\cdots,k\}} \bar{\mathcal{O}}_p^{i,j}$

Theorem 2 implies that there exists $q \in [k+2, \infty)$ s.t. $(x,r) \in \bar{\mathcal{O}}_q^{i,j}$

$$\Rightarrow (x(q-k-1; x,r), r) \in \bigcup_{p=0}^{k+1} \bar{\mathcal{O}}_p^{i,j} = \bigcup_{p=0}^{k} \bar{\mathcal{O}}_p^{i,j}$$

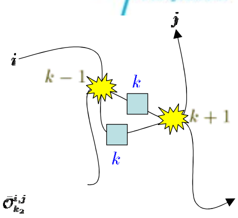
$$\Rightarrow (x(q-k-2; x,r), r) \in \bigcup_{p=0}^{k+1} \bar{\mathcal{O}}_p^{i,j} = \bigcup_{p=0}^{k} \bar{\mathcal{O}}_p^{i,j} \ \cdots\cdots\cdots$$

$$\Rightarrow (x,r) \in \bigcup_{p=0}^{k+1} \bar{\mathcal{O}}_p^{i,j} = \bigcup_{p=0}^{k} \bar{\mathcal{O}}_p^{i,j}$$ ⇒ contradicts $(x,r) \notin \bigcup_{p \in \{0,\cdots,k\}} \bar{\mathcal{O}}_p^{i,j}$

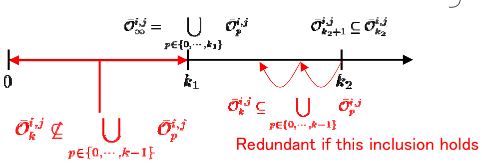
Redundant Polyhedra

About $\bar{\mathcal{O}}_{k+1}^{i,j} \subseteq \bar{\mathcal{O}}_k^{i,j} \Leftarrow \bar{\mathcal{O}}_\infty^{i,j} = \bigcup_{p \in \{0,\cdots,k\}} \bar{\mathcal{O}}_p^{i,j}$

This does not always hold true



Eliminations of redundant polyhedra



$$k^* := \min k \text{ s.t. } \bar{\mathcal{O}}_\infty^{i,j} = \bigcup_{p \in \{0,\cdots,k\}} \bar{\mathcal{O}}_p^{i,j}$$

Main Algorithm

Step 0: $k := 0$

$$\bar{\mathcal{O}}_0^{i,j} := \left\{ \begin{bmatrix} x \\ r \end{bmatrix} \middle| \ (x,r) \in \mathcal{S}_X \text{ and } x \in \bar{\mathbb{Y}}^{i,j} \right\}$$

Step 1: $\bar{\mathcal{O}}_{k+1}^{i,j} := \left\{ \begin{bmatrix} x \\ r \end{bmatrix} \middle| \ (x,r) \in \mathcal{S}_X \text{ and } Ax + Br \in \bar{\mathcal{O}}_k^{i,j} \right\}$

Step 2: Check if $\bar{\mathcal{O}}_{k+1}^{i,j} \subseteq \bar{\mathcal{O}}_k^{i,j}$

If it does not hold, then let $k := k+1$ and go to Step 1

Step 3: Eliminate redundant polyhedra

Step 4: Compute the SR

$$\bar{\mathcal{O}}_\infty^{i,j} = \bigcup_{p \in \{0,\cdots,k^*\}} \bar{\mathcal{O}}_p^{i,j} \Rightarrow \mathcal{O}_\infty^{i,j} = \bigcap_{p \in \{0,\cdots,k^*\}} \mathcal{O}_p^{i,j}$$

Union of convex polyhedra

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Analysis and Design

Analysis problem $\Omega_\infty(d) := \{x \mid \|x^i - x^j\|_\infty \le d \ \forall i,j\}$

- Distance which the collision is not avoidable

$$d_{\min} := \max_{d, x_v, i, j} d \text{ subject to } (x,r) \in \bar{O}_\infty^{i,j} \ \forall r \in \mathbb{R} \text{ and } x_p \in \Omega_\infty(d)$$

- Distance which the collision is avoidable (without taking account of the communication constraint)

$$d_{\max} := \max_{x,i,j} \|x_p^i - x_p^j\|_\infty \text{ subject to } (x,r) \in \bar{O}_\infty^{i,j} \ \forall r \in \mathbb{R}$$

Design problem

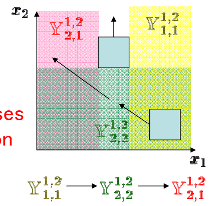
Computation of modified reference achieving the CA without taking into account the communication constraint

$$\min_g \|r - g\|_\infty \text{ subject to } (x,g) \in O_\infty^{i,j} \ \forall i,j$$

Problem

We have to consider all the pairs of regions both in analysis and design problems

Inherently, the computational effort increases exponentially and combinational optimization is inevitable

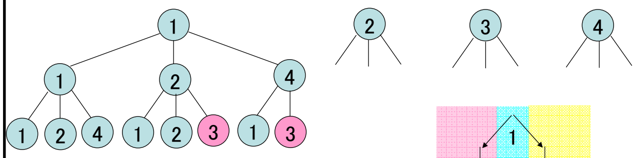


Keviczky et. al. (2006):Mixed Integer Program → NP hard
Kon et. al.(2007): Efficient branch and bound approach

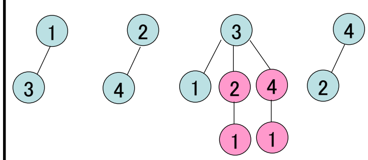
- Switching controllers based on heuristics
- Kon's approach + (Las Vegas Type) Randomized approach + Probabilistic rounding

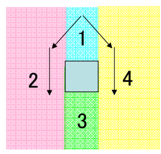
Hereafter, we consider only two agents (for simplicity)

Elimination of Redundant Scenarios



Events which never happen



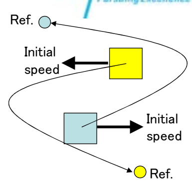


The number of switches depends on the dynamics and constraints. RHC can allows more complex switches, which leads to enormous computational burden

Elimination of Redundant Scenarios

Example of 5 switches

- Whether or not such an event happens depends on
  - Constraints (how fast initial state is admissible)
  - Attenuation rate of the zero input response
- Additionally, this motion restricts the behaviors after the corresponding time interval



4→1→2→3→4

$$4^{k^*} \Rightarrow \sum_{i=1}^{5} \binom{k^*}{i} \times 4 \times 3^{i-1}$$

$O(k^{*5})$

Polynomial order w.r.t. k^*

Can a certain switching pattern happen?

Find x, r subject to The constraint is satisfied and CA is avoided via this pattern

Linear constraint

Numerical Simulation

Dynamics of each agent: we consider only two agents

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0.1 & 0 \\ 1 & 1 & 0 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

This model is discretized with sampling period $0.1[s]$ by using zero-th order hold

Constraints: $\|x_p(t)\|_\infty \le 10, \|x_v(t)\|_\infty \le 1$ and $\|u(t)\|_\infty \le 10$

We design the optimal servo controller with the weighting matrices $Q = \text{diag}([10, 10, 1, 1, 10, 10]), R = I$

Parameter: $\epsilon = 0.2$

Simulation results:

$k^* = 12$

$k = 0 \sim 12$: Number of constraints $4^{13} = 67108864 \Rightarrow 3918$

CPU time: $6369.2[s]$

Conclusions

- **In this talk, we have presented**
  - an algorithm for computing the safe region
  - a method for reducing on-line computational effort
- **Future works will be directed to**
  - further reduction of on-line computational effort
  - extension to distributed environment
  - experimental validation
  - extension to distributed MPC

We also try to present switching controller strategies based on heuristics and set invariance theory

確率的丸め

整数線形計画問題

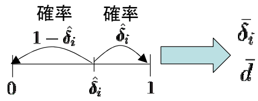
$$\min c^\top \delta \text{ subject to } \delta_i \in \{0,1\} \ \ \forall i \in \{1,\cdots,n\}$$

最適解 $\delta^*,\ J^* := c_d \delta^*$

線形計画問題（当然多項式時間！）

$$\min c^\top \delta \text{ subject to } \delta_i \in [0,1] \ \forall i \in \{1,\cdots n\}$$

最適解 $\hat{\delta},\ \hat{J} := c_d \hat{\delta}$



信頼度 $1-\beta$ で次式が成立する

$$\bar{d} \le \hat{d}(1+\epsilon) \le d^*(1+\epsilon)$$

ここで $\Pr\{d > (1+\epsilon)\hat{d}\} \le \beta/2n$

問題点：ラスベガス型アルゴリズムは解を返さないことがある

→ 前のステップと同じ修正目標値？