Coverage Control Problem with Anisotropic Sensors – part I

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Outline

• Coverage Problem Review (Proof)
• Problem Setting of Anisotropic case
• Anisotropic Voronoi Partition
• Future Works
Coverage Problem - Review

- **Objective**: Given agents \((p_1, \ldots, p_n)\), convex environment \(Q\), achieve optimal coverage.

- Let \(\phi\) be density function.

- Let \(f\) be sensing performance (non-decreasing)

\[ f(\|q - p_i\|) : \text{how poor } p_i \text{ to sense } q \]

- **Objective function**:

\[
H(P, W) = \int_{Q} f(\|q - p_i\|) \phi(q) dq = \sum_{i=1}^{n} \int_{w_i} f(\|q - p_i\|) \phi(q) dq
\]

Coverage Problem - Review

For $f(\|q - p_i\|) = \|q - p_i\|^2$ and dynamics $p_i = u_i$

$$u_i = -k(p_i - C_{v_i}) \quad (1)$$

Assume that $V(P) = \{V_1, \cdots, V_n\}$ is continuously updated

Proposition (Continuous-time Lloyd descent): For the closed loop system induced by (1), the sensors location converges asymptotically to the set of critical points of $H$, i.e., the set of centroidal Voronoi configurations on $Q$. Assuming this set is finite, the sensors location converges to a centroidal Voronoi configuration.

Coverage Problem - Review

Proof:

\[
\frac{\partial H}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q - p_i\|) \phi(q) dq
\]

\[
H(P) = -\sum_{i=1}^{n} \int_{V_i(P)} \|q - p_i\|^2 \phi(q) dq
\]

\[
\frac{\partial H}{\partial p_i}(P) = 2 \int_{V_i(P)} (q - p_i) \phi(q) dq = 2M_{V_i(P)} \left( CM_{V_i(P)} - p_i \right)
\]

\[
\frac{d}{dt} H(P(t)) = \sum_{i=1}^{n} \frac{\partial H_v}{\partial p_i} \dot{p}_i
\]

\[
= -2k \sum_{i=1}^{n} M_{V_i(P)} \left\| p_i - CM_{V_i(P)} \right\|^2
\]

- By LaSalle’s principle, the sensors location converges to the set of centroid Voronoi configurations.
If the set is a finite collection of points, then $P(t)$ converges to one of them

**LaSalle’s principle**: Let $\Omega \subset D$ be a compact set that it is positively invariant with respect to $X$. Let $x(0) \in M$ and $x_*$ be an accumulation point of $x(t)$. Then $x_* \in M$ and $\text{dist}(x(t), M) \to 0$ as $t \to \infty$

**Corollary**: If the set $M$ is a finite collection of points, then the limit of $x(t)$ exists and equals one of them
Coverage Problem - Review

\[ \frac{\partial H(P)}{\partial p_1} = \int_0^{p_1+p_2} \frac{\partial}{\partial p_1} \|q - p_1\|^2 dq \]

1 dimension 2 agents, \(Q=[0,1]\)

Proof:

\[ \frac{\partial H(P)}{\partial p_1} = \frac{\partial}{\partial p_1} \int_0^{p_1+p_2} \|q - p_1\|^2 dq + \frac{\partial}{\partial p_1} \int_{p_1/2}^{p_1+p_2} \|q - p_2\|^2 dq \]

\[ \frac{\partial}{\partial p_1} \int_0^{p_1+p_2} \|q - p_1\|^2 dq = \lim_{h \to 0} \frac{1}{h} \left( \int_0^{p_1+p_2} \|q - p_1 + h\|^2 dq - \int_0^{p_1+p_2} \|q - p_1\|^2 dq \right) \]

\[ = \lim_{h \to 0} \frac{1}{h} \left( \int_0^{p_1+h+p_2} \|q - p_1 + h\|^2 dq + \int_0^{p_1+p_2} \|q - p_1 + h\|^2 - \|q - p_1\|^2 dq \right) \]

Mean value theorem

\[ = \lim_{h \to 0} \frac{1}{h} \left( \int_0^{p_1+h+p_2} \|q - p_1 + h\|^2 dq + \int_0^{p_1+p_2} \frac{\partial}{\partial p_1} \|q - p_1\|^2 dq \right) \]

\[ = \lim_{h \to 0} \frac{1}{h} \|c - p_1 + h\|^2 \left( \frac{p_1 + h + p_2}{2} - \frac{p_1 + p_2}{2} \right), \exists c \in \left[ \frac{p_1 + p_2}{2}, \frac{p_1+p_2+h}{2} \right] \]

\[ = \lim_{h \to 0} \frac{1}{2} \|c - p_1 + h\|^2, \exists c \in \left[ \frac{p_1 + p_2}{2}, \frac{p_1+p_2+h}{2} \right] \]
Coverage Problem - Review

\[
\frac{1}{2} \left\| \frac{p_2 - p_1}{2} \right\|^2 + \int_0^{p_1 + p_2} \frac{\partial}{\partial p_1} \left\| q - p_1 \right\|^2 dq
\]

\[
\frac{\partial}{\partial p_1} \int_{p_1 + p_2}^{1/2} \left\| q - p_2 \right\|^2 dq = \lim_{h \to 0} \frac{1}{h} \left( \int_{p_1 + h + p_2}^{1/2} \left\| q - p_2 \right\|^2 dq - \int_{p_1 + p_2}^{1/2} \left\| q - p_2 \right\|^2 dq \right)
\]

\[
= -\lim_{h \to 0} \frac{1}{h} \left( \int_{p_1 + p_2}^{1/2} \left\| q - p_2 \right\|^2 dq \right)
\]

\[
= -\frac{1}{2} \lim_{h \to 0} \left\| c - p_2 \right\|^2, \exists c \in \left[ \frac{p_1 + p_2 + h}{2}, \frac{p_1 + p_2}{2} \right]
\]

\[
= -\frac{1}{2} \left\| \frac{p_2 - p_1}{2} \right\|^2
\]

\[
\therefore \frac{\partial H(P)}{\partial p_1} = \int_0^{p_1 + p_2} \frac{\partial}{\partial p_1} \left\| q - p_1 \right\|^2 dq
\]
Simulation-1

- Given 9 agents distributed in a square area of density $\Phi=1$
Simulation - 2

Case 1

position (y)

Case 2

position (x)

position (y)

position (x)

$H_1 > H_2$
Coverage - Anisotropic sensor

Goal: 3D Coverage Problem

First step: 2D Coverage Problem with Orientation of robots

Uniform (isotropic) sensor
- Sensing degradation only depends on distance

Non-uniform (anisotropic) sensor
- Sensing degradation depends on distance and orientation

Cortes et al.

Today’s talk: Uniform (isotropic) sensor - Sensing degradation only depends on distance.
Coverage – Anisotropic sensor

\[ \| q - p_i \|_L = (q - p_i)^T L (q - p_i) \]

\[
L = \begin{pmatrix}
\cos \theta_i & \sin \theta_i \\
-\sin \theta_i & \cos \theta_i \\
\end{pmatrix}
\begin{pmatrix}
\frac{c^2}{a^2} & 0 \\
0 & \frac{c^2}{b^2} \\
\end{pmatrix}
\begin{pmatrix}
\cos \theta_i & \sin \theta_i \\
-\sin \theta_i & \cos \theta_i \\
\end{pmatrix}
\]

\( \theta_i \) : orientation of \( i \)-th sensor

\( a, b \) : length of major and minor axis of the ellipse

Note:

If \( L = I \)  \( \implies \| q - p_i \| = (q - p_i)^T (q - p_i) \)

(Cortes et.al)
Problem Setting

Find $P, \Theta, W$ that minimize the function:

$$H(P, \Theta, W) = \sum_{i=1}^{n} \int_{W_i} f\left(\|q - p_i\|_L \Phi(q)\right) dq$$

Questions:

1. What is the Optimum Partition?
2. How do we optimize? One after another or at the same time?
Q: What is the optimum partition for a fixed sensors’ position and orientation?

- Anisotropic Voronoi partition: 
  \[ V_i^* = \left\{ q \in Q \mid \| q - p_i \|_L \leq \| q - p_j \|_L, \forall j \neq i \right\} \]

- Determined by sensor’s position and Orientation

Lemma: the boundary between two adjacent \( V_i^* \) and \( V_j^* \) is a quadratic curve
Anisotropic Voronoi Partition

Proof:
Any point $q$ in $V_i^* \cap V_j^*$ satisfies $\|q - p_i\|_L = \|q - p_j\|_L$
i.e. $(q - p_i)^T L_i (q - p_i) = (q - p_j)^T L_j (q - p_j)$. It is clear that this equation is quadratic in $q$. (Q.E.D)

Remark:

$$Ax^2 + By^2 + Cxy + Dx + Ey + K = 0,$$

$$A = b^2 (\cos^2 \theta_i - \cos^2 \theta_j) + a^2 (\sin^2 \theta_i - \sin^2 \theta_j)$$

$$B = a^2 (\cos^2 \theta_i - \cos^2 \theta_j) + b^2 (\sin^2 \theta_i - \sin^2 \theta_j)$$

$$C = (a^2 - b^2) (\sin 2\theta_i - \sin 2\theta_j)$$

If the sensors have the same orientation, the boundary will be a line.
Anisotropic Voronoi - Examples

boundary
Future Works

The optimization function:

\[ H(P, \Theta) = \sum_{i=1}^{n} \int_{v_i} f(\|q - p_i\|_L \Phi(q) dq) \]

- Fixed orientation case:
  1. what is the optimum position? (centroid of AVT?)
  2. Does \[ \frac{\partial H(P)}{\partial p_i} = \int_{v_i(P)} \frac{\partial}{\partial p_i} \|q - p_i\|^2 dq \] still hold?
- For fixed position case, what is the optimum orientation?
- Can we do fixed orientation opt. \(\rightarrow\) fixed position opt.?