Information based robot movement

Nico Hübel

Fujita Lab
Dept. of Mechanical and Control Engineering
Tokyo Institute of Technology

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Common assumptions:

Voronoi-based approaches

Information/Flocking-based approaches

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Voronoi-based approaches

- static after deployment
- huge amount of robots needed

Information/Flocking-based approaches

Common assumptions:

Voronoi-based approaches

- static after deployment
- huge amount of robots needed

Information/Flocking-based approaches

- converge to a fixed formation/flock
- then start tracking a target or to maximizing a none-decaying information measure

My Assumptions:

1.) Only a few robots

2.) Information decay

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It is not possible to see every point of interest continuously.

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1.) Only a few robots

It is not possible to see every point of interest continuously.

2.) Information decay

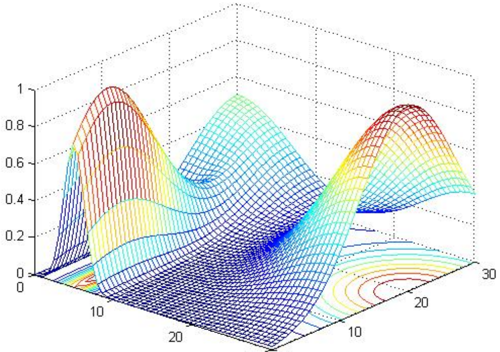
Every point must be revisited to gather new information.,

Problem formulation

1. Basic setup
2. Agent model
3. Measurement model
4. Information model
5. Objective function

Basic setup:

- $\mathcal{D}$: compact subset of $\mathbb{R}^2$ which should be covered
- $Q = \mathbb{R}^2$ configuration space of the Agents
- $\phi(q,t) : \mathcal{D} \times \mathbb{R}_0^+ \to \mathbb{R}^+$ positive definite density function



Agent model:

kinematic model:

$$\dot{\mathbf{q}}_i = \mathbf{u}_i(\mathbf{q}_i), \quad i \in S, \quad \mathbf{q}_i, \mathbf{u}_i(\mathbf{q}_i) \in \mathbb{R}^2 \tag{1}$$

- $\mathbf{q}_i \in Q$ position of Agent $\mathcal{A}_i$
- $\mathcal{A} = \{\mathcal{A}_i | i \in S = \{1, 2, 3, \ldots, N\}\}$ set of Agents
- $N \in \mathbb{R}^+$ number of Agents

Measurement model:

Measurement model:

measurement function: $\mathcal{M}_i\left(\|\mathbf{q}-\mathbf{q}_i\|^2\right) : \mathcal{D}\times Q \to \mathbb{R}_0^+$ (2)

measurement map: $\mathcal{M} = \sum_i \mathcal{M}_i, \quad i \in S$ (3)

- $\mathcal{M}_i \in \mathcal{C}^1$
- $\mathcal{M}_i(0) = M_i > \mathcal{M}_i(s), \forall s \neq 0$
- $\mathcal{W}_i = \{\mathbf{q} \in \mathcal{D} | s \leq r_i\}$
- $\mathcal{W} = \bigcup_{i\in S} \mathcal{W}_i$
- $\mathcal{M}_i(s) = 0 \forall s > r_i, \forall \mathbf{q} \in \mathcal{D} \setminus \mathcal{W}_i$

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Example:

$$\mathcal{M}_i = \begin{cases} \frac{M_i}{r_i^4}(s - r_i^2)^2 & \text{if } s \leq r_i \\ 0 & \text{if } s > r_i \end{cases}$$

Measurement model:

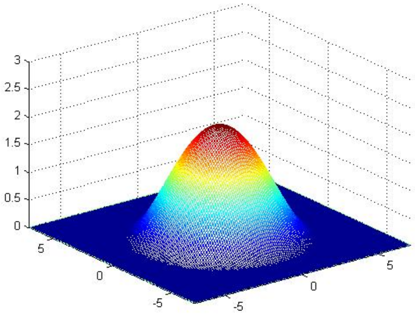

Figure: Example with $M_i = 2$ and $r_i = 5$.

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$$\mathcal{M}_i = \begin{cases} \frac{M_i}{r_i^4}(s - r_i^2)^2 & \text{if } s \leq r_i \\ 0 & \text{if } s > r_i \end{cases}$$

Information model:

Information model:

$$\frac{\partial}{\partial t} I(\mathbf{q}, t) = \delta I(\mathbf{q}, t) + \mathcal{M}(\mathbf{q}, \mathbf{q}_1, \dots, \mathbf{q}_N) \tag{4}$$

- $\delta \leq 0, \delta \in \mathbb{R}$ decay rate
- $\mathcal{M}$ measurement map
- $I : \mathcal{D} \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ information map
- $I_{max}(\mathbf{q}, t)$ information reference map

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Note:

Information model is a PDE!

Objective function:

Objective function:

$$J(t) = \int_{\mathcal{D}} h(I_{max}(\mathbf{q}, t) - I(\mathbf{q}, t))\phi(\mathbf{q}, t)d\mathbf{q} \tag{5}$$

- $h(x) \in \mathcal{C}^2$
- $h(x) : \mathbb{R} \rightarrow \mathbb{R}_0^+$
- $h(x), \frac{\partial h}{\partial x}(x), \frac{\partial^2 h}{\partial x^2}(x) \geq 0, \quad \forall x \neq 0$

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Example:

$h(x) = (max(0, x))^2$

Theorem:

1. Fully connected network
2. Partially connected network

Fully connected network:

Control law:

$$\mathbf{u}_i(\mathbf{q}_i) = -k_i \int_{\mathcal{D}} \frac{\partial h}{\partial x}\bigg|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)} \frac{\partial \mathcal{M}_i}{\partial s}\bigg|_{s=\|\mathbf{q}-\mathbf{q}_i\|^2} (\mathbf{q}-\mathbf{q}_i)\phi(\mathbf{q},t)d\mathbf{q} \quad (6)$$

$k_i \in \mathbb{R}^+$

Fully connected network:

Control law:

$$\mathbf{u}_i(\mathbf{q}_i) = -k_i \int_{\mathcal{D}} \frac{\partial h}{\partial x}\bigg|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)} \frac{\partial \mathcal{M}_i}{\partial s}\bigg|_{s=\|\mathbf{q}-\mathbf{q}_i\|^2} (\mathbf{q}-\mathbf{q}_i)\phi(\mathbf{q},t)d\mathbf{q} \quad (6)$$

$k_i \in \mathbb{R}^+$

Note:

- Integral evaluation over $\mathcal{W}_i$ instead over $\mathcal{D}$ sufficient! $(\frac{\partial \mathcal{M}_i}{\partial s} \neq 0$ only in $\mathcal{W}_i)$
- Input is zero if $I \geq I_{max} \quad \forall \mathbf{q} \in \mathcal{W}_i$!

Proof 1:

Assumption:

- fully connected network
- ϕ, I_{max} are time invariant

Take $J(t)$ as Lyapunov-function V:

$$V = J(t) = \int_{\mathcal{D}} \underbrace{h(I_{max}(\mathbf{q},t) - I(\mathbf{q},t))}_{\geq 0} \underbrace{\phi(\mathbf{q},t)}_{>0} \, d\mathbf{q} \geq 0$$

$$\Downarrow \frac{d}{dt} \text{and} (4)$$

$$\dot{V} = \frac{d}{dt} J(t) = \int_{\mathcal{D}} \underbrace{\frac{\partial h}{\partial x}\bigg|_{x = I_{max}(\mathbf{q},t) - I(\mathbf{q},t)}}_{\geq 0} \left(\underbrace{-\delta I(\mathbf{q},t)}_{\geq 0} \underbrace{-\mathcal{M}(\|\mathbf{q} - \mathbf{q}_i\|^2)}_{\leq 0} \right) \underbrace{\phi(\mathbf{q},t)}_{>0} \, d\mathbf{q} \overset{!}{\leq} 0 \tag{7}$$

Proof 1: (continue)

Rewrite (7):

$\left(\text{use abbreviations:} h' = \frac{\partial h}{\partial x}\Big|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)}, \quad h'' = \frac{\partial^2 h}{\partial x^2}\Big|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)}\right)$

$$0 \geq -\delta \int_{\mathcal{D}} h' I(\mathbf{q},t)\phi(\mathbf{q},t)d\mathbf{q} - \int_{\mathcal{D}} h' \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\phi(\mathbf{q},t)d\mathbf{q}$$

$$\Leftrightarrow (0 \geq)\delta \geq -\frac{\int_{\mathcal{D}} h' \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\phi(\mathbf{q},t)d\mathbf{q}}{\int_{\mathcal{D}} h' I(\mathbf{q},t)\phi(\mathbf{q},t)d\mathbf{q}} \quad (8)$$

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Interpretation:

- Bounds on δ!
- Nice physical interpretation

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Interpretation:

- Bounds on δ!
- Nice physical interpretation

Problem:

Lower bound on δ will be violated after a certain time!

Proof 1: (continue)

Now we look at the case when (8) is violated:
$-\delta \int_{\mathcal{D}} h' I(\mathbf{q},t)\phi(\mathbf{q},t)d\mathbf{q} \geq \int_{\mathcal{D}} h' \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\phi(\mathbf{q},t)d\mathbf{q}$

$$(7)$$

$$\Downarrow \frac{d}{dt} \text{and} (4), (3), (2), (1)$$

$$\frac{d^2}{dt^2} J(t) = \int_{\mathcal{D}} \underbrace{h'' \left(-\delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right)^2 \phi(\mathbf{q},t)}_{\geq 0} \, d\mathbf{q}$$

$$+ \int_{\mathcal{D}} \underbrace{\delta}_{\leq 0} \underbrace{h' \left(-\delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right) \phi(\mathbf{q},t)}_{\geq 0} \, d\mathbf{q}$$

$$+ 2 \int_{\mathcal{D}} \underbrace{h'}_{\geq 0} \sum_i \underbrace{\left.\frac{\partial \mathcal{M}_i}{\partial s}\right|_{s=\|\mathbf{q}-\mathbf{q}_i\|^2}}_{\leq 0} \underbrace{(\mathbf{q}-\mathbf{q}_i)\mathbf{u}_i}_{\geq 0} \underbrace{\phi(\mathbf{q},t)}_{\geq 0} \, d\mathbf{q} \overset{!}{\leq} 0$$

Proof 1: (continue)

With (6):

$$\frac{d^2}{dt^2}J(t) = \int_{\mathcal{D}} h'' \left(-\delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right)^2 \phi(\mathbf{q},t)d\mathbf{q}$$
$$+ \int_{\mathcal{D}} \delta h' \left(-\delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right) \phi(\mathbf{q},t)d\mathbf{q}$$
$$- 2\sum_i k_i \left(\int_{\mathcal{D}} h' \left.\frac{\partial \mathcal{M}_i}{\partial s}\right|_{s=\|\mathbf{q}-\mathbf{q}_i\|^2} (\mathbf{q}-\mathbf{q}_i)\phi(\mathbf{q},t)d\mathbf{q}\right)^2 \leq 0$$

Proof 1: (continue)

With (6):

$$\frac{d^2}{dt^2}J(t) = \int_{\mathcal{D}} h'' \left(-\delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right)^2 \phi(\mathbf{q},t)d\mathbf{q} + \int_{\mathcal{D}} \delta h' \left(-\delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right) \phi(\mathbf{q},t)d\mathbf{q} - 2\sum_i k_i \left(\int_{\mathcal{D}} h' \left.\frac{\partial \mathcal{M}_i}{\partial s}\right|_{s=\|\mathbf{q}-\mathbf{q}_i\|^2} (\mathbf{q}-\mathbf{q}_i)\phi(\mathbf{q},t)d\mathbf{q}\right)^2 \leq 0$$

Problems:

- How to choose k_i?
- Integral evaluation over $\mathcal{W}_i$ instead over $\mathcal{D}$ sufficient! $(\frac{\partial \mathcal{M}_i}{\partial s} \neq 0$ only in $\mathcal{W}_i)$
- Input is zero if $I \geq I_{max} \quad \forall \mathbf{q} \in \mathcal{W}_i$!

Symmetry breaking:

Problem:

$I \geq I_{max} \quad \forall \mathbf{q} \in \mathcal{W}_i \Rightarrow h' = 0 \quad \forall \mathbf{q} \in \mathcal{W}_i \Rightarrow \mathbf{u}_i = 0$

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Problem:

$I \geq I_{max} \quad \forall \mathbf{q} \in \mathcal{W}_i \Rightarrow h' = 0 \quad \forall \mathbf{q} \in \mathcal{W}_i \Rightarrow \mathbf{u}_i = 0$

Solution:

Apply simple linear controller for symmetry breaking:

$$\hat{\mathbf{u}}_i(\mathbf{q}_i) = -\hat{k}_i \left(\mathbf{q}_i(t) - \hat{\mathbf{q}}_i(t_s) \right) \tag{9}$$

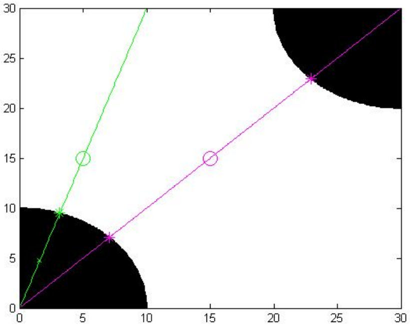
$\hat{k}_i \in \mathbb{R}^+, \quad t_s$: time at which symmetry conditions are fulfilled

$\hat{\mathbf{q}}_i(t_s)$: arbitrary point which leads to a non symmetric state

Symmetry breaking:

Example:

$\hat{\mathbf{q}}_i(t_s) \in \mathcal{D}^i_{ns}(t_s) = \left\{\tilde{\mathbf{q}} \in \mathcal{D}_{ns}(t_s) : \tilde{\mathbf{q}} = \arg\min_{\mathbf{q}\in\mathcal{D}_{ns}(t_s)} \|\mathbf{q} - \mathbf{q}_i\|\right\}$
with $\mathcal{D}_{ns}(t_s) = \{\mathbf{q} \in \mathcal{D} : I(\mathbf{q}, t) < I_{max}(\mathbf{q})\}$



Theorem:

Theorem 1:

Under the given assumptions the control law

$$\mathbf{u}_i^*(\mathbf{q}_i) = \begin{cases} \mathbf{u}_i(\mathbf{q}_i) & \text{if } h' \neq 0 \quad \text{for some } \mathbf{q} \in \mathcal{W}_i \\ \hat{\mathbf{u}}_i(\mathbf{q}_i) & \text{if } h' = 0 \quad \forall \mathbf{q} \in \mathcal{W}_i \end{cases} \tag{10}$$

with sufficiently large gains $k_i,\ \hat{k}_i \in \mathbb{R}^+$ will minimize the objective function and thus maximize the information over the area $\mathcal{D}$.

Partially connected network:

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Worst case scenario:

In principle the centralized algorithm holds for $\mathcal{A} = \mathcal{A}_1$. Thus theorem 1 holds for a swarm of disconnected robots or several disconnected networks!

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Worst case scenario:

In principle the centralized algorithm holds for $\mathcal{A} = \mathcal{A}_1$. Thus theorem 1 holds for a swarm of disconnected robots or several disconnected networks!

Estimated objective function:

$$\hat{J}_i(t) = \int_{\mathcal{D}} h(I_{max}(\mathbf{q},t) - I_{\hat{S}_i}(\mathbf{q},t))\phi(\mathbf{q},t)d\mathbf{q} \tag{11}$$

with $\hat{S}_i \subseteq S$ as the index set of those agents of which $\mathcal{A}_i$ receives information.

Proof 2:

Note:

$\hat{J}_i(t) \geq J(t)$ with equality holding iff $\hat{S}_i = S$.

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Assumption:

Bidirectional communication:
if $j \in \hat{S}_i$ then $i \in \hat{S}_j$

Proof 2:

Note:

$\hat{J}_i(t) \geq J(t)$ with equality holding iff $\hat{S}_i = S$.

Assumption:

Bidirectional communication:
if $j \in \hat{S}_i$ then $i \in \hat{S}_j$

Proof 2:

With $\hat{V} = \hat{J}_i(t)$ we get similar expressions to those obtained in proof 1. Thus the rest of proof 2 follows exactly proof 1!

Future work:

- incorporate other robot dynamics
- think about the assumption on $\mathcal{M}_i$ and $h(x)$
- extend problem to $Q = \mathbb{R}^3$ or $Q = SE(2)$
- use (Q,S,R)-Dissipativity for proof
- incorporate time-varying density function and/or information reference map

Derivation of the equations for proof 1:

Starting from eqn. (5):

$$J(t) = \int_{\mathcal{D}} h(I_{max}(\mathbf{q},t) - I(\mathbf{q},t))\phi(\mathbf{q},t)d\mathbf{q}$$

$$\Downarrow \frac{d}{dt}$$

$$\frac{d}{dt}J(t) = \int_{\mathcal{D}} \left(\left.\frac{\partial h}{\partial x}\right|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)} \left(\frac{\partial}{\partial t} I_{max}(\mathbf{q},t) - \frac{\partial}{\partial t} I(\mathbf{q},t) \right) \phi(\mathbf{q},t) + h\left(I_{max}(\mathbf{q},t) - I(\mathbf{q},t)\right) \frac{\partial}{\partial t}\phi(\mathbf{q},t) \right) d\mathbf{q}$$

$$\stackrel{(4)}{=} \int_{\mathcal{D}} \left(\left.\frac{\partial h}{\partial x}\right|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)} \left(\frac{\partial}{\partial t} I_{max}(\mathbf{q},t) - \delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2) \right) \phi(\mathbf{q},t) + h\left(I_{max}(\mathbf{q},t) - I(\mathbf{q},t)\right) \frac{\partial}{\partial t}\phi(\mathbf{q},t) \right) d\mathbf{q}$$

$$\Downarrow \frac{d}{dt}$$

.

Derivation of the equations for proof 1: (continue)

$$\vdots$$

$$\Downarrow \frac{d}{dt}$$

$$\begin{aligned}
\frac{d^2}{dt^2}J(t) = \int_{\mathcal{D}} \Bigg(& \frac{\partial^2 h}{\partial x^2}\bigg|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)} \left(\frac{\partial}{\partial t} I_{max}(\mathbf{q},t) - \delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right)^2 \phi(\mathbf{q},t) \\
& + \frac{\partial h}{\partial x}\bigg|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)} \left(\frac{\partial^2}{\partial t^2} I_{max}(\mathbf{q},t) - \delta^2 I(\mathbf{q},t) - \delta\mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right) \phi(\mathbf{q},t) \\
& - \frac{\partial h}{\partial x}\bigg|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)} \underbrace{\frac{d}{dt}\mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)}_{\overset{(3),(2),(1)}{=} -2\sum_i \frac{\partial \mathcal{M}_i}{\partial s}\big|_{s=\|\mathbf{q}-\mathbf{q}_i\|^2}(\mathbf{q}-\mathbf{q}_i)\mathbf{u}_i} \phi(\mathbf{q},t) \\
& + 2\,\frac{\partial h}{\partial x}\bigg|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)} \left(\frac{\partial}{\partial t} I_{max}(\mathbf{q},t) - \delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right) \frac{\partial}{\partial t}\phi(\mathbf{q},t) \\
& + h(I_{max}(\mathbf{q},t) - I(\mathbf{q},t))\frac{\partial^2}{\partial t^2}\phi(\mathbf{q},t)\Bigg)\, d\mathbf{q}
\end{aligned}$$

Derivation of the equations for proof 1: (continue)

Assumption: ϕ, I_{max} are time invariant

$$J(t) = \int_{\mathcal{D}} h(I_{max}(\mathbf{q},t) - I(\mathbf{q},t))\phi(\mathbf{q},t)d\mathbf{q}$$

$$\frac{d}{dt}J(t) = \int_{\mathcal{D}} \left(\left.\frac{\partial h}{\partial x}\right|_{x=I_{max}(\mathbf{q},t)-I(\mathbf{q},t)} \left(-\delta I(\mathbf{q},t) - \mathcal{M}(\|\mathbf{q}-\mathbf{q}_i\|^2)\right)\phi(\mathbf{q},t) \right) d\mathbf{q}$$

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