# Passivity-based Attitude Coordination with Weighted Graph



Passivity-based Attitude Coordination with Weighted Graph

FL07-05-1

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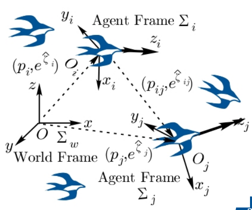
## Outline



1. About Cooperative Control
2. Attitude Coordination with Weighted Graph
3. Connectivity Analysis
4. Switching Topology
5. Feature Works

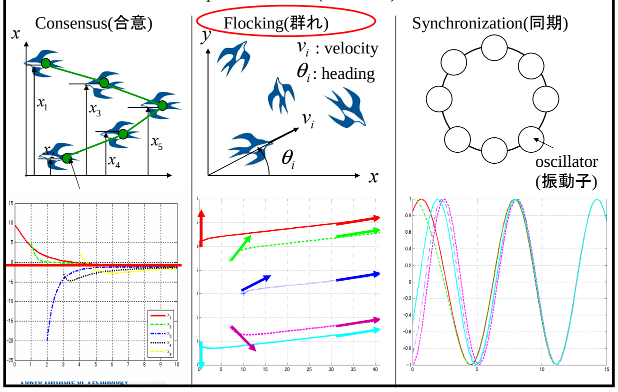





## Cooperative Control



Cooperative Control(協調制御)




## Synchronization



http://physics.owu.edu/StudentResearch/2005/BryanDaniels/intro.html

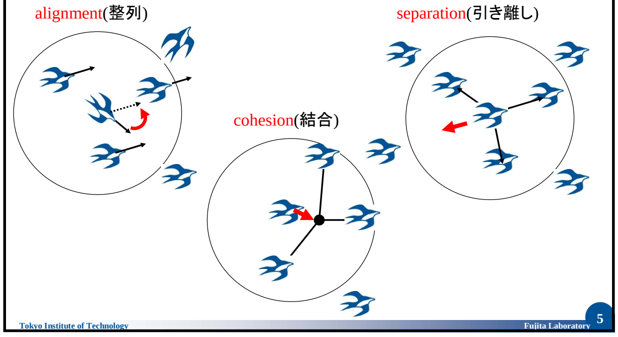


## Flocking: Raynolds Model



Craig Raynolds 1987

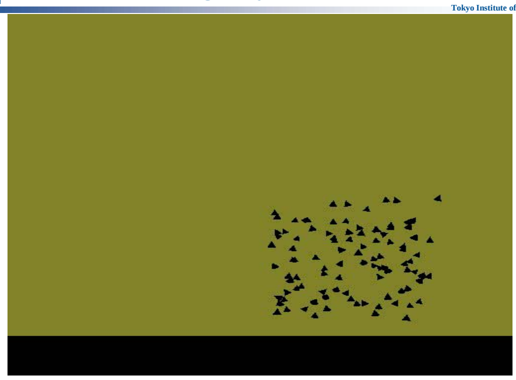
Three simple local rules for "boids"






## Flocking: Raynolds Model



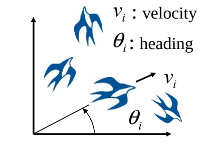


http://angel.elte.hu/~vicsek/

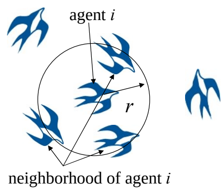
# Passivity-based Attitude Coordination with Weighted Graph

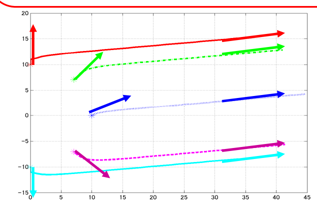
## Flocking: Vicsek Model

Vicsek 1995




$$\theta_i(k+1) = \tan^{-1}\frac{\left(\sum_{j\in N_i}\sin\theta_j(k)\right)+\sin\theta_i(k)}{\left(\sum_{j\in N_i}\cos\theta_j(k)\right)+\cos\theta_i(k)}$$

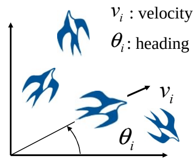




## Flocking

A. Jadbabaie, J. Lin and A. S. Morse 2003




$$\theta_i(k+1) = \tan^{-1}\frac{\left(\sum_{j\in N_i}\sin\theta_j(k)\right)+\sin\theta_i(k)}{\left(\sum_{j\in N_i}\cos\theta_j(k)\right)+\cos\theta_i(k)}$$

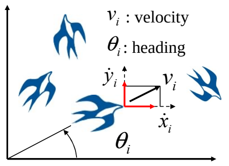
For small angles
$$\theta_i(t+1) = \frac{1}{1+n_i(t)}\left(\theta_i(t)+\sum_{j\in N_i}\theta_j(t)\right)$$

Theorem(Jadbabaie et al. 2003)
If there is a sequence of bounded, non-overlapping time intervals $T_k$ such that over any interval of length $T_k$ ,the network of agents is "jointly connected", then all agents will reach consensus on their velocity vectors



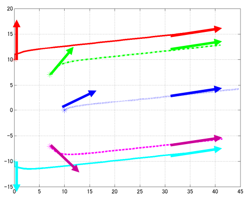
## Flocking

N. Moshtagh and A. Jadbabaie 2005




Dynamics of agent $i$(continuous time)

$$\begin{cases}\dot{x}_i = v\cos\theta_i \\ \dot{y}_i = v\sin\theta_i \\ \dot{\theta}_i = u_i\end{cases}$$
$$u_i = -\sum_{j\in N_i}\sin(\theta_i-\theta_j)$$



$$\dot{\theta}_i = -\sum_{j\in N_i}\sin(\theta_i-\theta_j)$$

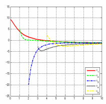
$$-\frac{\pi}{2}<\theta_i<\frac{\pi}{2} \qquad i=1,\cdots,n$$



## Cooperative Control

Consensus

$$\dot{x}_i = -\sum_{j\in N_i}(x_i-x_j)$$



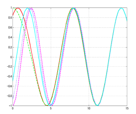
Flocking
Discrete time model

$$\theta_i(k+1) = \tan^{-1}\frac{\left(\sum_{j\in N_i}\sin\theta_j(k)\right)+\sin\theta_i(k)}{\left(\sum_{j\in N_i}\cos\theta_j(k)\right)+\cos\theta_i(k)}$$

N. Moshtagh, et al CDC,2006
Continuous time model
$$\dot{\theta}_i = -\sum_{j\in N_i}\sin(\theta_i-\theta_j)$$

Synchronization

$$\dot{\theta}_i = \omega - \frac{K}{N}\sum_{j\in N_i}\sin(\theta_i-\theta_j)$$



$$\dot{x}_i = \sum_{j\in N_i}(x_j - <x_j,x_i> x_i)$$
$$x_i = \begin{bmatrix}\cos\theta_i \\ \sin\theta_i\end{bmatrix}$$

R. O. Saber ,CDC,2006
<>: inner product

$$\theta_i = \phi_i - \omega$$

3D case(SE(3))
Attitude Coordination
My research

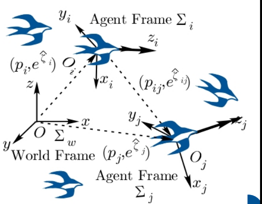
generalize

Output Synchronization
N. Chopra, M. W. Spong



## Outline







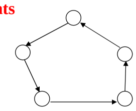

## Previous Results

- **Goal** Attitude Coordination
$$\lim_{t\to\infty}(e^{\hat{\xi}_i(t)} - e^{\hat{\xi}_j(t)}) = 0$$
- **Control Input** $\omega_i^b = \sum_{j\in N_i}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$ There is no input gains
- **Analysis** (07 1/29 FL semi, 07 Bumon)

Assumptions
- Graph is balanced, fixed and strongly connected.
- At the initial time $t=0$ , the agents' orientation matrices, $e^{\hat{\xi}_i(0)}\forall i$ are positive definite

Potential Function sum of energy functions of each agents

$$V = \sum_{i=1}^{n}\phi(e^{\hat{\xi}_i}) := \sum_{i=1}^{n}\frac{1}{2}\mathrm{tr}(I_3 - e^{\hat{\xi}_i})$$



Key points
$$\sum_{i=1}^{n}\sum_{j\in N_i}\phi(e^{\hat{\xi}_i}) = \sum_{i=1}^{n}\sum_{j\in N_i}\phi(e^{\hat{\xi}_j})$$
$$-\mathrm{tr}\left((e^{\hat{\xi}_i}+e^{-\hat{\xi}_i})(I-e^{-\hat{\xi}_i}e^{\hat{\xi}_j})\right) \le -\lambda_{\min}\left(e^{\hat{\xi}_i}+e^{-\hat{\xi}_i}\right)\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})$$

# Passivity-based Attitude Coordination with Weighted Graph

## Today's Presentation

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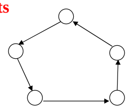
- **Analysis** **(07 1/29 FL semi, 07 Bumon)**

Assumptions
- Graph is balanced, fixed and strongly connected.
- At the initial time $t=0$, the agents' orientation matrices, $e^{\hat{\xi}_i(0)}\forall i$ are positive definite

Potential Function **sum of energy functions of each agents**

$$V=\sum_{i=1}^{n}\phi(e^{\hat{\xi}_i}):=\sum_{i=1}^{n}\frac{1}{2}\mathrm{tr}(I_3-e^{\hat{\xi}_i})$$



**(Today's Presentation)**

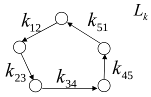
- **Control Input** $\omega_i^b=\sum_{j\in N_i}k_{ij}\,\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee \quad k_{ij}>0$
- **Convergence Analysis** Potential Function

$$V=\sum_{i=1}^{n}\gamma_i\phi(e^{\hat{\xi}_i}):=\sum_{i=1}^{n}\frac{1}{2}\gamma_i\mathrm{tr}(I_3-e^{\hat{\xi}_i})$$

Weighted Graph Laplacian

$$L_k=\{L_{kij}\}=\begin{cases}\sum_{j\in N_i}k_{ij} & \text{if } i=j\\ -k_{ij} & \text{if } j\in N_i\\ 0 & \text{if } j\notin N_i\end{cases}$$

- **Connectivity Analysis**
- **Switching Topology**

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## Problem Statement

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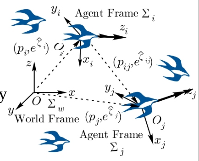
- **Agent Model** $(i=1,\cdots,n)$

$$\dot{p}_i=e^{\hat{\xi}_i}v_i \qquad \dot{\varsigma}_i=e^{\hat{\xi}_i}\omega_i^b \quad (1) \qquad \varsigma_i=\theta_i\xi_i$$

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}^\wedge:=\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}$$

- $p_i\in\mathcal{R}^3$ position
- $e^{\hat{\xi}_i}\in SO(3)$ orientation
- $v_i\in\mathcal{R}^3$ body velocity
- $\omega_i^b\in\mathcal{R}^3$ angular velocity
- $\theta_i\in\mathcal{R}$ rotation angle
- $\xi_i\in\mathcal{R}^3$ rotation axes




If

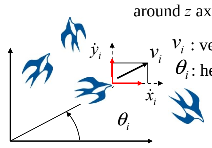
$$e^{\hat{\xi}_i}=\begin{bmatrix}\cos\theta_i & -\sin\theta_i & 0\\ \sin\theta_i & \cos\theta_i & 0\\ 0 & 0 & 1\end{bmatrix},\quad v_i=\begin{bmatrix}v\\0\\0\end{bmatrix}\ \forall i$$

(It's means all agents rotate only around $z$ axis)

then (1) is equal to

$$\begin{cases}\dot{x}_i=v\cos\theta_i\\ \dot{y}_i=v\sin\theta_i\\ \dot{\theta}_i=\omega_i\end{cases}$$

(cf. N. Moshtagh and A. Jadbabaie 2005)

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## Goal

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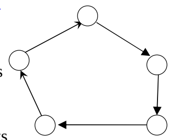
- **Using a graph to represent the Intersection topology**

Graph $G$ : Graph consist of a pair $(V(G),E(G))$, where $V(G)$ is a finite nonempty set of nodes and $E(G)\subseteq V(G)\times V(G)$ is a set of pair of nodes ,called edges



$G:=(V,E)$ : Graph

$V:=\{1,\cdots n\}$ : A set of vertices indexed by set of agents

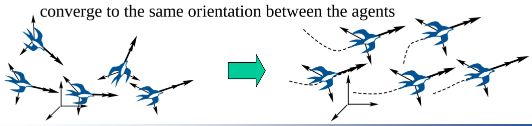
$E\subseteq V\times V$ : A set of edges the represent the neighboring relations

neighborhood $N_i$ : A set of agents whose information is available to agent $i$

- **Goal** Attitude Coordination

$$\lim_{t\to\infty}(e^{\hat{\xi}_i(t)}-e^{\hat{\xi}_j(t)})=0\quad\forall i,j$$

A group of agents is said to Attitude Coordination,when all agents converge to the same orientation between the agents



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## Control Input

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- **Control Input**

$$\omega_i^b=\sum_{j\in N_i}k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee \quad (2) \quad k_{ij}>0$$

Previous result

$$\omega_i^b=\sum_{j\in N_i}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})$$

$e^{-\hat{\xi}_i}e^{\hat{\xi}_j}$ : Relative orientation

$N_i$ : Agent $i$'s neighborhood

$$\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}^\vee=\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$$

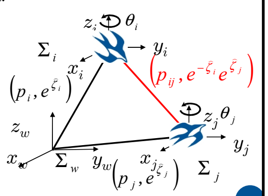
If $k_{ij}=1\ \forall i,j$ ,then experiments weren't successful.
So, I add the input gain to my previous result.

If

$$e^{\hat{\xi}_i}=\begin{bmatrix}\cos\theta_i & -\sin\theta_i & 0\\ \sin\theta_i & \cos\theta_i & 0\\ 0 & 0 & 1\end{bmatrix},\ k_{ij}=1,\ \forall i,j$$

then $\omega_i^b=\sum_{j\in N_i}k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$ is equal to

$$\omega_i^b=-\sum_{j\in N_i}\sin(\theta_i-\theta_j)$$

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## Assumptions

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- <u>Assumptions</u> (A)

(A1) At the initial time $t=0$, the agents' orientation matrices, $e^{\hat{\xi}_i(0)}\ \forall i$ are positive definite

(A2) $|v_i|=1\ \forall i$ each agent's speed is constant and normalized.

(A3) Graph is ~~balanced,~~ fixed and strongly connected.

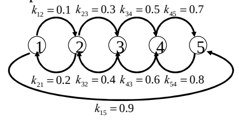
(A4) Elements of left eigenvector of the following matrix associated with eigenvalue 0 can be positive.

$$L_k=\{L_{kij}\}=\begin{cases}\sum_{j\in N_i}k_{ij} & \text{if } i=j\\ -k_{ij} & \text{if } j\in N_i\\ 0 & \text{if } j\notin N_i\end{cases}$$

Weighted Graph Laplacian

i.e. $\gamma^T L_k=0^T \quad \gamma^T=[\gamma_1\ \cdots\ \gamma_n] \quad \gamma_i>0 \quad \forall i$

Example




$$L_k=\begin{bmatrix}0.1 & -0.1 & 0 & 0 & 0\\ -0.2 & 0.5 & -0.3 & 0 & 0\\ 0 & -0.4 & 0.9 & -0.5 & 0\\ 0 & 0 & -0.6 & 1.3 & -0.7\\ -0.9 & 0 & 0 & -0.8 & 1.7\end{bmatrix}\quad \gamma=\begin{bmatrix}0.9318\\0.3138\\0.1593\\0.0821\\0.0338\end{bmatrix}$$

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## Assumptions

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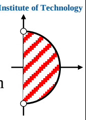
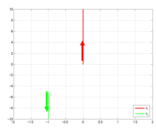
- **Remark**

(A1) Rotation matrices $e^{\hat{\xi}_i}$ are positive definite if and only if $|\theta_i|<\frac{\pi}{2}$.
Maintaining the condition in assumption (A1) avoids the problem of singularity of orientation.

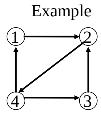


Example(singular case)

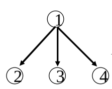


(A4) This assumption is satisfied if graph is not balanced.
It means this assumption is milder than previous results.

Example



$$L_k=\begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\\ -1 & -1 & 0 & 2\end{bmatrix}\quad \gamma=\begin{bmatrix}0.3780\\0.3780\\0.7559\\0.3780\end{bmatrix}$$



$$L_k=\begin{bmatrix}0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}\quad \gamma=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

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## Passivity



Definition ([1] pp.236) The system $\dot{x} = f(x,u)$, $y = h(x,u)$ is said to be passive. If there exists a continuously differentiable positive semidefinite function $V(x)$ (called the storage function) such that

$$u^T y \geq \dot{V} \quad \forall (x,u) \in R^n \times R^p$$

[1] H. Khalil, *Nonlinear Systems*, 2002

In this case, the following equation is satisfied.

Lemma Consider the agents given by (1). Then the following equation hold for the each agent.

$$\nu_i^T y_i = \dot{V}_i$$

where $\nu_i^T := [v_i^T \ \omega_i^T]^T$, $y_i^T := [(e^{\hat{\xi}_i} p_i)^T \ (\mathrm{sk}(e^{\hat{\xi}_i})^\vee)^T]^T$ and $V_i := \frac{1}{2}\|p_i\|^2 + \phi(e^{\hat{\xi}_i})$

$\phi(e^{\hat{\xi}_i}) := \frac{1}{2}\mathrm{tr}(I_3 - e^{\hat{\xi}_i})$ $I_3$: Identity Matrix

*Proof*: This lemma can be easily proven by direct calculation of the derivative of the positive definite function $V_i$.



## Convergence Analysis



- **Convergence Analysis**

Define the potential function as the sum of orientation term of positive definite function $V_i$. This potential function is the same previous result's potential function.

$$V = \sum_{i=1}^{n} \phi(e^{\hat{\xi}_i}) = \frac{1}{2}\sum_{i=1}^{n} \mathrm{tr}(I_3 - e^{\hat{\xi}_i}) \qquad I_3: \text{Identity Matrix}$$

The derivative of this potential function along trajectories of the system (1)

$$\dot{V} = \sum_{i=1}^{n} \dot{\phi}(e^{\hat{\xi}_i}) = \sum_{i=1}^{n} \left(\mathrm{sk}(e^{\hat{\xi}_i})^\vee\right)^T \underline{\omega_i^b} \qquad \underline{\omega_i^b} = \sum_{j \in N_i} k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j})^\vee \text{ is given as}$$

$$= \sum_{i=1}^{n} \left(\mathrm{sk}(e^{\hat{\xi}_i})^\vee\right)^T \sum_{j \in N_i} k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j})^\vee$$

Using $a^T b = -\frac{1}{2}\mathrm{tr}(\hat{a}\hat{b})$, we can show

$$= -\sum_{i=1}^{n}\sum_{j \in N_i} k_{ij}\phi(e^{\hat{\xi}_i}) + \sum_{i=1}^{n}\sum_{j \in N_i} k_{ij}\phi(e^{\hat{\xi}_j})$$

$$-\frac{1}{4}\sum_{i=1}^{n}\sum_{j \in N_i} k_{ij}\mathrm{tr}\left((e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})(I - e^{-\hat{\xi}_i}e^{\hat{\xi}_j})\right)$$



## Convergence Analysis



$$= \underline{-\sum_{i=1}^{n}\sum_{j \in N_i} k_{ij}\phi(e^{\hat{\xi}_i}) + \sum_{i=1}^{n}\sum_{j \in N_i} k_{ij}\phi(e^{\hat{\xi}_j})}$$

$$-\frac{1}{4}\sum_{i=1}^{n}\sum_{j \in N_i} k_{ij}\mathrm{tr}\left((e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})(I - e^{-\hat{\xi}_i}e^{\hat{\xi}_j})\right)$$

If graph is balanced and $k_{ij} = 1 \forall i,j$ (= previous result)

$$\underline{\sum_{i=1}^{n}\sum_{j \in N_i} \phi(e^{\hat{\xi}_i}) - \sum_{i=1}^{n}\sum_{j \in N_i} \phi(e^{\hat{\xi}_j}) = 0}$$

but $\sum_{i=1}^{n}\sum_{j \in N_i} k_{ij}\phi(e^{\hat{\xi}_i})$ is not always equal to $\sum_{i=1}^{n}\sum_{j \in N_i} k_{ij}\phi(e^{\hat{\xi}_j})$.

So we must redesign the potential function.



## Theorem1



Theorem 1 Consider the system $n$ agents with mode given by (1). Under the assumptions (A), the control input (2) achieves attitude coordination. Namely $\lim_{t \to \infty}(e^{\hat{\xi}_i} - e^{\hat{\xi}_j}) = 0 \quad \forall i,j$

***Sketch of Proof*** (Please see appendix1 in detail)

Define the potential function as the following function

$$V = \sum_{i=1}^{n} \gamma_i \phi(e^{\hat{\xi}_i}) := \sum_{i=1}^{n} \frac{1}{2}\gamma_i \mathrm{tr}(I_3 - e^{\hat{\xi}_i})$$

$\gamma$ is a left eigenvector of the weighted graph Laplacian

$$\dot{V} = \sum_{i=1}^{n} \gamma_i \dot{\phi}(e^{\hat{\xi}_i}) = \sum_{i=1}^{n} \gamma_i \left(\mathrm{sk}(e^{\hat{\xi}_i})^\vee\right)^T \underline{\omega_i^b} \qquad \underline{\omega_i^b} = \sum_{j \in N_i} k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j})^\vee$$

$$= \sum_{i=1}^{n} \gamma_i \left(\mathrm{sk}(e^{\hat{\xi}_i})^\vee\right)^T \sum_{j \in N_i} k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i} e^{\hat{\xi}_j})^\vee$$

Using $a^T b = -\frac{1}{2}\mathrm{tr}(\hat{a}\hat{b})$, we can show

$$= -\sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\phi(e^{\hat{\xi}_i}) + \sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\phi(e^{\hat{\xi}_j})$$

$$-\frac{1}{4}\sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\mathrm{tr}\left((e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})(I - e^{-\hat{\xi}_i}e^{\hat{\xi}_j})\right)$$



## Theorem1



Equation

$$-\sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\phi(e^{\hat{\xi}_i}) + \sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\phi(e^{\hat{\xi}_j})$$

can be changed to

$$-\gamma^T L_k \Phi = 0 \qquad \gamma^T L_k = 0^T \quad L_k = \{L_{kij}\} = \begin{cases} \sum_{j \in N_i} k_{ij} & \text{if } i = j \\ -k_{ij} & \text{if } i \in N_j \\ 0 & \text{if } i \notin N_i \end{cases} \qquad \Phi := \begin{bmatrix} \phi(e^{\hat{\xi}_1}) \\ \vdots \\ \phi(e^{\hat{\xi}_n}) \end{bmatrix}$$

So

$$-\sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\phi(e^{\hat{\xi}_i}) + \sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\phi(e^{\hat{\xi}_j}) = 0$$

Consequently

$$\dot{V} = \sum_{i=1}^{n} \gamma_i \dot{\phi}(e^{\hat{\xi}_i}) = -\frac{1}{4}\sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\mathrm{tr}\left((e^{\hat{\xi}_i} + e^{-\hat{\xi}_i})(I - e^{-\hat{\xi}_i}e^{\hat{\xi}_j})\right)$$

Now rotation matrices $e^{\hat{\xi}_i} \ \forall i$ are assumed to be positive definite, therefore they satisfy the following inequality

$$\lambda_{\min}(B + B^T)\mathrm{tr}(A) \leq \mathrm{tr}((B + B^T)A) \leq \lambda_{\max}((B + B^T))\mathrm{tr}(A)$$

Therefore the derivative of the potential function reduces to

$$\dot{V} \leq -\frac{1}{2}\sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\lambda_{\min}\left(e^{\hat{\xi}_i} + e^{-\hat{\xi}_i}\right)\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}) \leq 0$$



## Theorem1



Using LaSalle's Invariance Principle

$$0 = \dot{V} \leq -\frac{1}{2}\sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\lambda_{\min}\left(e^{\hat{\xi}_i} + e^{-\hat{\xi}_i}\right)\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}) \leq 0$$

$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j \in N_i} \gamma_i k_{ij}\lambda_{\min}\left(e^{\hat{\xi}_i} + e^{-\hat{\xi}_i}\right)\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}) = 0$$

$$\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}) = 0 \quad (i,j) \in E \quad \gamma_i > 0 \quad k_{ij} > 0 \quad \lambda_{\min}\left(e^{\hat{\xi}_i} + e^{-\hat{\xi}_i}\right) > 0$$

Now the graph is assumed strongly connected, so

$$\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}) = 0 \quad (i,j) \in E \quad \Rightarrow \quad \phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}) = 0 \quad \forall i,j$$

$\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}) = 0$ means orientation of every $i$-th agent converge to the same value

**Each agents converge to the same orientation**

Q.E.D.

# Passivity-based Attitude Coordination with Weighted Graph

## Time delay case

The results can be extended to the case when are communication delay. In this case by attitude coordination we mean the following

- **Goal** **Attitude Coordination** (time delay case)

$$\lim_{t\to\infty}(e^{\hat{\xi}_i(t-T_{ij})}-e^{\hat{\xi}_j(t)})=0 \qquad \forall i,j$$

$T_{ji}$ : sum of the delays along the path from the agent $i$ to the agent $j$.

- **Control Input**

$$\omega_i=\sum_{j\in N_i}k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i(t)}e^{\hat{\xi}_j(t-T_{ij})})^{\vee}$$

Attitude coordination in the above sense can by shown using the positive definite Lyapunov-Krasovskii function.

$$V_{delay}:=\sum_{i=1}^{n}\gamma_i\phi(e^{\hat{\xi}_i(t)})+\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\int_{t-T_{ji}}^{t}\phi(e^{\hat{\xi}_i(\tau)})d\tau$$

The proof is similar to no time delay case. (Please see appendix2 in detail)



## Simulation

- **Simulation (no time delay)**

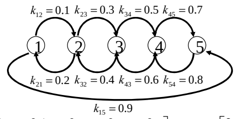
$$p_1(0)=\begin{pmatrix}1\\0\\3\end{pmatrix}\quad \zeta_1(0)=\begin{pmatrix}-0.21\\-0.50\\0.77\end{pmatrix}$$

$$p_2(0)=\begin{pmatrix}2\\-1\\2\end{pmatrix}\quad \zeta_2(0)=\begin{pmatrix}0.52\\-0.76\\0.52\end{pmatrix}$$

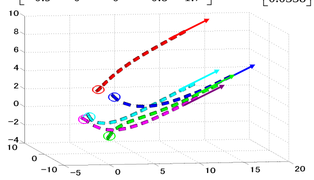
$$p_3(0)=\begin{pmatrix}3\\1\\2\end{pmatrix}\quad \zeta_3(0)=\begin{pmatrix}-0.21\\0.77\\-0.50\end{pmatrix}$$

$$p_4(0)=\begin{pmatrix}-1\\-2\\0\end{pmatrix}\quad \zeta_4(0)=\begin{pmatrix}-0.14\\0.51\\-0.51\end{pmatrix}$$

$$p_5(0)=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad \zeta_5(0)=\begin{pmatrix}0\\1.05\\0\end{pmatrix}$$

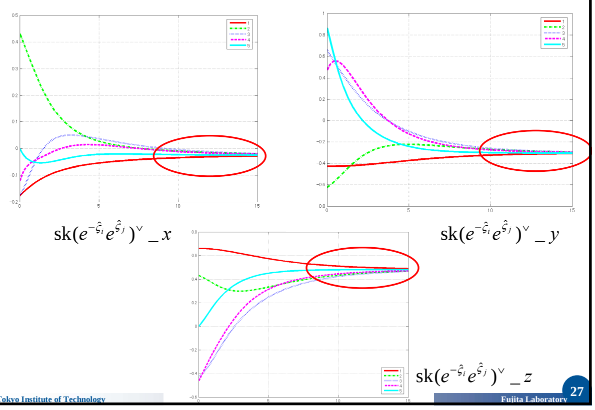



$$L_k=\begin{bmatrix}0.1&-0.1&0&0&0\\-0.2&0.5&-0.3&0&0\\0&-0.4&0.9&-0.5&0\\0&0&-0.6&1.3&-0.7\\-0.9&0&0&-0.8&1.7\end{bmatrix}\qquad \gamma=\begin{bmatrix}0.9318\\0.3138\\0.1593\\0.0821\\0.0338\end{bmatrix}$$





## Simulation



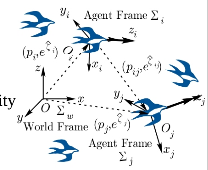
$\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^{\vee}\_x$ $\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^{\vee}\_y$ $\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^{\vee}\_z$



## Summary

- **Agent Model** $(i=1,\cdots,n)$

$$\dot{p}_i=e^{\hat{\varsigma}_i}v_i \qquad p_i\in\mathcal{R}^3 \text{ position}$$
$$\varsigma_i=e^{\hat{\varsigma}_i}\omega_i^b \quad (1) \qquad e^{\hat{\varsigma}_i}\in SO(3) \text{ orientation}$$
$$\varsigma_i=\theta_i\xi_i \qquad v_i\in\mathcal{R}^3 \text{ body velocity}$$
$$\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\end{bmatrix}^{\wedge}:=\begin{bmatrix}0&-\omega_3&\omega_2\\\omega_3&0&-\omega_1\\-\omega_2&\omega_1&0\end{bmatrix} \qquad \omega_i^b\in\mathcal{R}^3 \text{ angular velocity},\ \theta_i\in\mathcal{R} \text{ rotation angle},\ \xi_i\in\mathcal{R}^3 \text{ rotation axes}$$




- **Control Input** $\omega_i^b=\sum_{j\in N_i}k_{ij}\mathrm{sk}(e^{-\hat{\varsigma}_i}e^{\hat{\varsigma}_j})^{\vee}$ (2) $k_{ij}>0$

- **Assumptions**

At the initial time $t=0$, the agents' orientation matrices, $e^{\hat{\varsigma}_i(0)}\ \forall i$ are positive definite

$\gamma^T L_k=0^T$ $\gamma^T=[\gamma_1\ \cdots\ \gamma_n]$ $\gamma_i>0$ $\forall i$

$L_k$ : Weighted Graph Laplacian

- **Potential Function**

$$V=\sum_{i=1}^{n}\gamma_i\phi(e^{\hat{\varsigma}_i}):=\sum_{i=1}^{n}\frac{1}{2}\gamma_i\mathrm{tr}(I_3-e^{\hat{\varsigma}_i})$$



## Outline

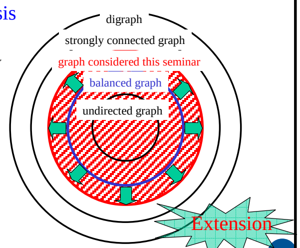
✓ 1. About Cooperative Control

✓ 2. Attitude Coordination with weighted graph

3. Connectivity Analysis

4. Switching Topology

5. Feature works





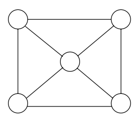

## Connectivity Analysis

- **Graph Structure and Convergence Speed**
  - R. O. Saber, J. A. Fax and R. M. Murray, Proc. IEEE, 95-1, 2007

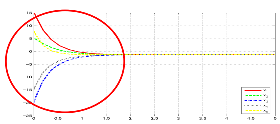
Consensus case

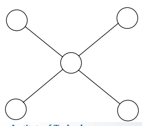
The second smallest eigenvalue of graph Laplacians $\lambda_{\min 2}(L)$, called the algebraic connectivity, quantifies the speed of convergence.

Example

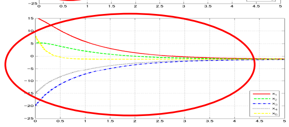


$\lambda_{\min 2}(L)=3$





$\lambda_{\min 2}(L)=1$



# Passivity-based Attitude Coordination with Weighted Graph

## Assumptions



In connectivity analysis, assumptions are changed to the following.

- **Assumptions** (B)

(B1) At the initial time $t=0$, the relative orientation matrices, $e^{-\hat{\xi}_i(0)}e^{\hat{\xi}_i(0)}\ \forall i,j$ are positive definite

(B2) $|v_i|=1\ \forall i$ each agent's speed is constant and normalized.

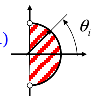
(B3) Graph is fixed and strongly connected.

(B4) Elements of left eigenvector of the following matrix associated with eigenvalue 0 can be positive.

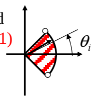
$$L_k=\{L_{kij}\}=\begin{cases}\sum_{j\in N_i}k_{ij} & \text{if } i=j\\ -k_{ij} & \text{if } j\in N_i\\ 0 & \text{if } j\notin N_i\end{cases}$$ Weight Graph Laplacian

i.e. $\gamma^T L_k=0^T \quad \gamma^T=[\gamma_1 \ \cdots \ \gamma_n] \quad \gamma_i>0 \quad \forall i$

region satisfied assumption (A1)




region satisfied assumption (B1)






## Theorem2



Theorem 2 Consider the $n$ agents (1) together with the angular velocity given by (2). Under the assumptions (B), $k_{ij}=1\ \forall i,j$ and graph is balanced, the following inequality is satisfied.

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\le \mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\lambda_{\min 2}(L_{sym})\varepsilon t}$$

$\lambda_{\min 2}(L_{sym})$: second minimum eigenvalue of $L_{sym}$

where

$$\left(e^{\hat{\xi}}\right)^T=\left[\left(e^{\hat{\xi}_1}\right)^T \ \cdots \ \left(e^{\hat{\xi}_n}\right)^T\right]\in R^{3n\times 3} \quad \varepsilon:=\min_{i,j,t}\lambda_{\min}\left(e^{-\hat{\xi}_i(t)}e^{\hat{\xi}_j(t)}+e^{-\hat{\xi}_j(t)}e^{\hat{\xi}_i(t)}\right) \quad \vec{1}:=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$

$L_{sym}:=\frac{1}{2}(L+L^T) \quad M=nI-\vec{1}\vec{1}^T$ : graph Laplacian of a complete graph

***Sketch of Proof***

Define the potential function as the following function

$$\frac{1}{2}\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)=\sum_{i=1}^n\sum_{k=1}^n\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})$$



## Theorem2



The derivative of the potential function is given as

$$\frac{d}{dt}\left(\frac{1}{2}\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\right)=\sum_{i=1}^n\sum_{k=1}^n\dot{\phi}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}) \qquad \omega_i^b=\sum_{j\in N_i}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$$

$$=\sum_{i=1}^n\sum_{k=1}^n\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^\vee\right)^T\left(-\omega_i^b+\omega_i^b\right)$$

$$=-2\sum_{i=1}^n\sum_{k=1}^n\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^\vee\right)^T\sum_{j\in N_i}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$$

Using $a^Tb=-\frac{1}{2}\mathrm{tr}(\hat{a}\hat{b})$, we can show

$$=\frac{1}{2}\sum_{i=1}^n\sum_{k=1}^n\sum_{j\in N_i}\mathrm{tr}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-I\right)\right)$$

Because of

$$\sum_{i=1}^n\sum_{k=1}^n\sum_{j\in N_i}\mathrm{tr}\left(e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-I\right)\right)=\sum_{i=1}^n\sum_{k=1}^n\sum_{j\in N_i}\mathrm{tr}\left(e^{-\hat{\xi}_k}e^{\hat{\xi}_j}-e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)=0$$

the above equation can be rewrite

$$=\frac{1}{2}\sum_{i=1}^n\sum_{k=1}^n\sum_{j\in N_i}\mathrm{tr}\left(\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}+e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-I\right)\right)$$



## Theorem2



Now rotation matrices $e^{\hat{\xi}_i}\ \forall i$ are assumed to be positive definite, therefore they satisfy the following inequality

$$\lambda_{\min}\left(B+B^T\right)\mathrm{tr}(A)\le\mathrm{tr}\left(\left(B+B^T\right)A\right)\le\lambda_{\max}\left(\left(B+B^T\right)\right)\mathrm{tr}(A)$$

$$\frac{d}{dt}\left(\frac{1}{2}\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\right)\le-\frac{1}{2}\sum_{i=1}^n\sum_{k=1}^n\sum_{j\in N_i}\lambda_{\min}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}+e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)\mathrm{tr}\left(I-e^{-\hat{\xi}_i}e^{\hat{\xi}_j}\right)$$

$$\le-\frac{1}{2}\varepsilon n\,\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(L\otimes I_3)e^{\hat{\xi}}\right) \qquad \left(\because \varepsilon:=\min_{i,j,t}\lambda_{\min}\left(e^{-\hat{\xi}_i(t)}e^{\hat{\xi}_j(t)}+e^{-\hat{\xi}_j(t)}e^{\hat{\xi}_i(t)}\right)\right)$$

$$=-\frac{\varepsilon}{2n}\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(ML_{sym}M\otimes I_3)e^{\hat{\xi}}\right) \qquad (\because MLM=n^2L)$$

$$\le-\frac{\varepsilon}{2}\lambda_{\min 2}(L_{sym})\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)e^{\hat{\xi}}\right)$$

Using Comparison Principle ([1] pp.102), we can show

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\le\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\lambda_{\min 2}(L_{sym})\varepsilon t}$$

Q.E.D.



## Theorem2'



If $k_{ij}>0$, the following matrix is important.

$$L_w=\{L_{wij}\}=\begin{cases}\sum_{j\in N_i}\gamma_i k_{ij} & \text{if } i=j\\ -\gamma_i k_{ij} & \text{if } j\in N_i\\ 0 & \text{if } j\notin N_i\end{cases} \qquad L_w=diag(\gamma_1,\cdots,\gamma_n)L_k$$

Properties of $L_w$

- $L_w\vec{1}=\vec{0}$ $\quad\left(\because L_w\vec{1}=diag(\gamma_1,\cdots,\gamma_n)L_k\vec{1}=\vec{0}\right)$
- $\vec{1}^TL_w=\vec{0}^T$ $\quad\left(\because \vec{1}^TL_w=\vec{1}^T diag(\gamma_1,\cdots,\gamma_n)L_k=\vec{0}^T\right)$
- $ML_wM=n^2L_w$ $\quad\left(\begin{aligned}\because ML_wM&=(nI-\vec{1}\vec{1}^T)L_w(nI-\vec{1}\vec{1}^T)\\&=(nI-\vec{1}\vec{1}^T)nL_w=n^2L_w\end{aligned}\right)$
- $L_w$ is semipositive definite



## Theorem2'



If $k_{ij}>0$ ,The theorem2 is changed to the theorem2'

Theorem 2 Consider the $n$ agents (1) together with the angular velocity given by (2). Under the assumptions (B), the following inequality is satisfied.

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\le\frac{\max_{i,j,i\neq j}\gamma_i\gamma_j}{\min_{i,j,i\neq j}\gamma_i\gamma_j}\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\frac{\varepsilon\lambda_{\min 2}(L_{wsym})}{n\max_{i,j,i\neq j}\gamma_i\gamma_j}\sum_{i=1}^n\gamma_i t}$$

$\lambda_{\min 2}(L_{sym})$ : second minimum eigenvalue of $L_{sym}$

where

$$\left(e^{\hat{\xi}}\right)^T=\left[\left(e^{\hat{\xi}_1}\right)^T \ \cdots \ \left(e^{\hat{\xi}_n}\right)^T\right]\in R^{3n\times 3} \quad \varepsilon:=\min_{i,j,t}\lambda_{\min}\left(e^{-\hat{\xi}_i(t)}e^{\hat{\xi}_j(t)}+e^{-\hat{\xi}_j(t)}e^{\hat{\xi}_i(t)}\right) \quad \vec{1}:=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$

$L_{wsym}:=\frac{1}{2}(L_{wsym}+L_{wsym}^T) \quad M=nI-\vec{1}\vec{1}^T$ : graph Laplacian of complete graph

***Sketch of Proof*** (Please see appendix3 in detail)

Define the potential function as the following function

$$\sum_{i=1}^n\sum_{k=1}^n\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})$$

$$\min_{i,k,i\neq k}\gamma_i\gamma_k\sum_{i=1}^n\sum_{k=1}^n\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\le\sum_{i=1}^n\sum_{k=1}^n\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\le\max_{i,k,i\neq k}\gamma_i\gamma_k\sum_{i=1}^n\sum_{k=1}^n\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})$$

# Passivity-based Attitude Coordination with Weighted Graph

## Theorem2’



The derivative of potential function is given as

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\dot{\phi}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})=\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^{\vee}\right)^T\left(-\underline{\omega_i^b}+\underline{\omega_k^b}\right)$$

$$=-2\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^{\vee}\right)^T\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^{\vee}\qquad \underline{\omega_i^b}=\sum_{j\in N_i}k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^{\vee}$$

Using $a^Tb=-\frac{1}{2}\mathrm{tr}(\hat{a}\hat{b})$ , we can show

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-I\right)\right)$$

Because of $\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-I\right)\right)=\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_k}e^{\hat{\xi}_j}-e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)=0$

the above equation can be rewrite

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}+e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-I\right)\right)$$

$$\leq-\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\lambda_{\min}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}+e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)\mathrm{tr}\left(I-e^{-\hat{\xi}_i}e^{\hat{\xi}_j}\right)$$



## Theorem2’



$$\leq-\frac{1}{2}\varepsilon\sum_{k=1}^{n}\gamma_k\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(I-e^{-\hat{\xi}_i}e^{\hat{\xi}_j}\right)\quad\left(\because\varepsilon:=\min_{i,j,t}\lambda_{\min}\left(e^{-\hat{\xi}_i(t)}e^{\hat{\xi}_j(t)}+e^{-\hat{\xi}_j(t)}e^{\hat{\xi}_i(t)}\right)\right)$$

$$=\frac{1}{2}\frac{\varepsilon}{n^2}\sum_{k=1}^{n}\gamma_k\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T\left(ML_{sym}M\otimes I_3\right)e^{\hat{\xi}}\right)\quad\left(\because ML_{wsym}M=n^2L_{wsym}\right)$$

$$\leq\frac{1}{2}\frac{\varepsilon}{n^2}\sum_{k=1}^{n}\gamma_k\lambda_{\min2}\left(L_{wsym}\right)\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)(M\otimes I_3)e^{\hat{\xi}}\right)$$

$$\frac{d}{dt}\left(\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\right)\leq-\frac{1}{\max\limits_{i,k,i\neq k}\gamma_i\gamma_k}\varepsilon\sum_{k=1}^{n}\gamma_k\left(L_{sym}\right)\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})$$

Using Comparison Principle ([1] pp.102), we can show

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\leq\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})e^{-\frac{\varepsilon}{n}\frac{\lambda_{\min2}(L_{wsym})}{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}\sum_{i=1}^{n}\gamma_i t}$$

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\leq\frac{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}{\min\limits_{i,j,i\neq j}\gamma_i\gamma_j}\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\frac{\varepsilon}{n}\frac{\lambda_{\min2}(L_{wsym})}{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}\sum_{i=1}^{n}\gamma_i t}$$

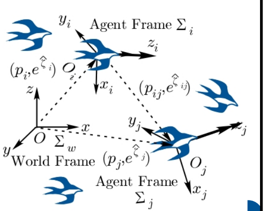
Q.E.D.



## Outline










## Switching Topology



We consider the situation where the information graph changes in time so as to make the graph alternatively connected and disconnected. Disconnections mean there are communication failures.

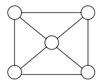
Definition new notations

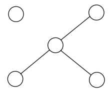
$G_c$ : A set of connected graph (stable situation)

$G_{dc}$ : A set of disconnected graph (unstable situation)

$t_k\ (k=0,\cdots,m)$ : Switching time

$N_G(\tau,t)$ : Switching times between $\tau$ and t



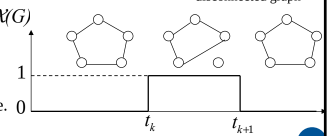




Definition new functions

$$\mathcal{X}(G):=\begin{cases}0 & \text{if } g(t)\in G_c\\ 1 & \text{if } g(t)\in G_{dc}\end{cases}$$

$$T(\tau,t):=\int_{\tau}^{t}\mathcal{X}(g(s))ds$$

$T(\tau,t)$ means unstable situation time.






## Assumptions



Assumptions are changed to the following.

- **Assumptions** (C)

(C1) At the initial time $t=0$ , the relative orientation matrices, $e^{-\hat{\varsigma}_i}e^{\hat{\varsigma}_i}$ $\forall i,j$ are positive definite

(C2) $|v_i|=1\ \forall i$ each agent’s speed is constant and normalized.

~~(C3) Graph is fixed and strongly connected.~~

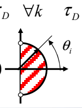
(C3)

$$L_k=\{L_{kij}\}=\begin{cases}\sum_{j\in N_i}k_{ij} & \text{if } i=j\\ -k_{ij} & \text{if } i\in N_j\\ 0 & \text{if } i\notin N_i\end{cases}\qquad \text{Weight Graph Laplacian}$$

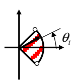
i.e. $\gamma^TL_k=0^T$ $\gamma^T=[\gamma_1\ \cdots\ \gamma_n]$ $\gamma_i>0\ \forall i$

(C4) $t_k-t_{k-1}\geq\tau_D\ \forall k$ $\tau_D>0$ $\tau_D$ : dwell time

region satisfied assumption (A1)




region satisfied assumption (C1)






## Theorem3



**Theorem 3** Consider the $n$ agents (1) together with the angular velocity given by (2). Under the assumptions (C), the following inequality is satisfied.

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\leq\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\lambda^*\left(t-T(0,t)-\frac{N_G(0,t)}{\lambda^*}\ln\gamma^*\right)}$$

where

$$\lambda^*:=\min_G\frac{\varepsilon}{n}\frac{\lambda_{\min2}(L_{wsym})}{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}\sum_{i=1}^{n}\gamma_i\qquad\gamma^*:=\max_G\frac{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}{\min\limits_{i,j,i\neq j}\gamma_i\gamma_j}$$

***Sketch of Proof***

From theorem2, if $t\in[t_{i-1},t_i)$

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\leq\begin{cases}\gamma^*\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\lambda^*(t-t_{i-1})} & \text{If } g(t)\in G_c\\ \gamma^*\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right) & \text{If } g(t)\in G_{dc}\end{cases}$$

If the graph is disconnected, rank of the graph Laplacian is below n-1.

$$\mathrm{rank}(L)\leq n-1$$

## Theorem3



Using $\mathcal{X}(G)$, the equation can be rewrite as the following.

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T (M\otimes I_3)e^{\hat{\xi}(t)}\right)\le \gamma^* \mathrm{tr}\left(\left(e^{\hat{\xi}(t_{i-1})}\right)^T (M\otimes I_3)e^{\hat{\xi}(t_{i-1})}\right)e^{-\lambda^*(1-\mathcal{X}(g(t))(t-t_{i-1})}$$

$$=\gamma^* \mathrm{tr}\left(\left(e^{\hat{\xi}(t_{i-1})}\right)^T (M\otimes I_3)e^{\hat{\xi}(t_{i-1})}\right)e^{-\lambda^*(t-t_{i-1})+\lambda^* T(t_{i-1},t)}$$

In $t\in[t_{i-2},t_{i-1})$

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t_{i-1})}\right)^T (M\otimes I_3)e^{\hat{\xi}(t_{i-1})}\right)\le \gamma^* \mathrm{tr}\left(\left(e^{\hat{\xi}(t_{i-2})}\right)^T (M\otimes I_3)e^{\hat{\xi}(t_{i-2})}\right)e^{-\lambda^*(t_{i-1}-t_{i-2})+\lambda^* T(t_{i-2},t_{i-1})}$$

So

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T (M\otimes I_3)e^{\hat{\xi}(t)}\right)\le \left(\gamma^*\right)^2 \mathrm{tr}\left(\left(e^{\hat{\xi}(t_{i-2})}\right)^T (M\otimes I_3)e^{\hat{\xi}(t_{i-2})}\right)e^{-\lambda^*(t-t_{i-2})+\lambda^* T(t_{i-2},t)}$$

Using this operation continuously, we can show

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T (M\otimes I_3)e^{\hat{\xi}(t)}\right)\le \left(\gamma^*\right)^{N_G(0,t)} \mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T (M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\lambda^* t+\lambda^* T(0,t)}$$



## Theorem3



$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T (M\otimes I_3)e^{\hat{\xi}(t)}\right)\le \left(\gamma^*\right)^{N_G(0,t)} \mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T (M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\lambda^* t+\lambda^* T(0,t)}$$

$$=\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T (M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\lambda^*\left(t-N_G(0,t)\frac{\ln(\gamma^*)}{\lambda^*}-T(0,t)\right)}$$

Q.E.D.

- **Remark**

From this theorem, if

$$\lim_{t\to\infty}\left(t-N_G(0,t)\frac{\ln\left(\gamma^*\right)}{\lambda^*}-T(0,t)\right)=\infty$$

then each agents' orientation converge to the same value, even if the graph is changed



## Feature Works



1. Experiments
2. Extension to Visual Attitude Coordination
3. Output Synchronization in SE(3)
4. Connection this result and game theory or MPC

⋮

and so on.



## Appendix



1. Proof of Theorem1
2. Proof of Time Delay Case
3. Proof of Theorem2'



## Proof of Theorem1



1. Proof of Theorem1



## Proof of Theorem1



Theorem 1 Consider the system $n$ agents with mode given by (1). Under the assumptions (A), the control input (2) achieves attitude coordination. Namely $\lim_{t\to\infty}(e^{\hat{\xi}_i}-e^{\hat{\xi}_j})=0 \quad \forall i,j$

***Proof***

Define the potential function as the following function

$$V=\sum_{i=1}^{n}\gamma_i\phi(e^{\hat{\xi}_i}):=\sum_{i=1}^{n}\frac{1}{2}\gamma_i \mathrm{tr}(I_3-e^{\hat{\xi}_i})$$

$$\dot{V}=\sum_{i=1}^{n}\gamma_i\dot{\phi}(e^{\hat{\xi}_i})=\sum_{i=1}^{n}\gamma_i\left(\mathrm{sk}(e^{\hat{\xi}_i})^\vee\right)^T\omega_i^b \quad \omega_i=\sum_{j\in N_i}k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$$

$$=\sum_{i=1}^{n}\gamma_i\left(\mathrm{sk}(e^{\hat{\xi}_i})^\vee\right)^T\sum_{j\in N_i}k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$$

$$=\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\left(\mathrm{sk}(e^{\hat{\xi}_i})^\vee\right)^T\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$$

## Proof of Theorem1



Using $a^T b = -\frac{1}{2}\mathrm{tr}(\hat a \hat b)$ , we can show

$$= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(\left(\mathrm{sk}(e^{\hat\zeta_i})\right)\mathrm{sk}(e^{-\hat\zeta_i}e^{\hat\zeta_j})\right)$$

$$= -\frac{1}{8}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(\left(e^{\hat\zeta_i} - e^{-\hat\zeta_i}\right)\left(e^{-\hat\zeta_i}e^{\hat\zeta_j} - e^{\hat\zeta_i}e^{-\hat\zeta_j}\right)\right)$$

$$= -\frac{1}{8}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(e^{\hat\zeta_i}e^{-\hat\zeta_i}e^{\hat\zeta_j} - e^{\hat\zeta_i}e^{\hat\zeta_i}e^{-\hat\zeta_j} - e^{-\hat\zeta_i}e^{-\hat\zeta_i}e^{\hat\zeta_j} + e^{-\hat\zeta_i}e^{\hat\zeta_i}e^{-\hat\zeta_j}\right)$$

$$= -\frac{1}{8}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(2e^{\hat\zeta_j} - 2e^{-2\hat\zeta_i}e^{\hat\zeta_j}\right) \qquad \left(\because \mathrm{tr}\left(e^{-\hat\zeta_i}\right) = \mathrm{tr}\left(e^{\hat\zeta_i}\right)\right)$$

$$= -\frac{1}{4}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(2I - 2e^{\hat\zeta_i} + 2e^{\hat\zeta_j} - 2I + (e^{\hat\zeta_i} + e^{-\hat\zeta_i})(I - e^{-\hat\zeta_i}e^{\hat\zeta_j})\right)$$

$$= -\frac{1}{4}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(e^{\hat\zeta_j} - e^{-2\hat\zeta_i}e^{\hat\zeta_j} + I - I + e^{-\hat\zeta_i} - e^{\hat\zeta_i}\right)$$

$$= -\frac{1}{4}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(I - e^{\hat\zeta_i} + e^{\hat\zeta_j} - I + e^{-\hat\zeta_i}(I - e^{-\hat\zeta_i}e^{\hat\zeta_j})\right)$$



## Proof of Theorem1



$$= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(I - e^{\hat\zeta_i}\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left(I - e^{\hat\zeta_j}\right)$$

$$-\frac{1}{4}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left((e^{\hat\zeta_i} + e^{-\hat\zeta_i})(I - e^{-\hat\zeta_i}e^{\hat\zeta_j})\right)$$

$$= -\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\phi(e^{\hat\zeta_i}) + \sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\phi(e^{\hat\zeta_j})$$

$$-\frac{1}{4}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left((e^{\hat\zeta_i} + e^{-\hat\zeta_i})(I - e^{-\hat\zeta_i}e^{\hat\zeta_j})\right)$$

Equation

$$-\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\phi(e^{\hat\zeta_i}) + \sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\phi(e^{\hat\zeta_j})$$

can be changed to

$$-\gamma^T L_k \Phi = 0$$

$$L_k = \{L_{kij}\} = \begin{cases} \sum_{j\in N_i} k_{ij} & \text{if } i = j \\ -k_{ij} & \text{if } i \in N_j \\ 0 & \text{if } i \notin N_i \end{cases} \quad \Phi := \begin{bmatrix} \phi(e^{\hat\zeta_1}) \\ \vdots \\ \phi(e^{\hat\zeta_n}) \end{bmatrix}$$

So

$$-\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\phi(e^{\hat\zeta_i}) + \sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\phi(e^{\hat\zeta_j}) = 0$$



## Proof of Theorem1



Consequently

$$\dot V = \sum_{i=1}^{n}\gamma_i\dot\phi(e^{\hat\zeta_i}) = -\frac{1}{4}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\mathrm{tr}\left((e^{\hat\zeta_i} + e^{-\hat\zeta_i})(I - e^{-\hat\zeta_i}e^{\hat\zeta_j})\right)$$

Now rotation matrices $e^{\hat\zeta_i}\ \forall i$ are assumed to be positive definite, therefore they satisfy the following inequality

$$\lambda_{\min}\left(B + B^T\right)\mathrm{tr}(A) \le \mathrm{tr}\left(\left(B + B^T\right)A\right) \le \lambda_{\max}\left(\left(B + B^T\right)\right)\mathrm{tr}(A)$$

Therefore the derivative of the potential function reduces to

$$\dot V \le -\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\lambda_{\min}\left(e^{\hat\zeta_i} + e^{-\hat\zeta_i}\right)\phi(e^{-\hat\zeta_i}e^{\hat\zeta_j}) \le 0$$

Using LaSalle's Invariance Principle

$$0 = \dot V \le -\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\lambda_{\min}\left(e^{\hat\zeta_i} + e^{-\hat\zeta_i}\right)\phi(e^{-\hat\zeta_i}e^{\hat\zeta_j}) \le 0$$

$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\lambda_{\min}\left(e^{\hat\zeta_i} + e^{-\hat\zeta_i}\right)\phi(e^{-\hat\zeta_i}e^{\hat\zeta_j}) = 0$$

$$\phi(e^{-\hat\zeta_i}e^{\hat\zeta_j}) = 0 \qquad (i,j)\in E \qquad \gamma_i > 0 \qquad k_{ij} > 0 \qquad \lambda_{\min}\left(e^{\hat\zeta_i} + e^{-\hat\zeta_i}\right) > 0$$



## Proof of Theorem1



Now the graph is assumed strongly connected, so

$$\phi(e^{-\hat\zeta_i}e^{\hat\zeta_j}) = 0 \quad (i,j)\in E \quad \Longrightarrow \quad \phi(e^{-\hat\zeta_i}e^{\hat\zeta_j}) = 0 \quad \forall i,j$$

$\phi(e^{-\hat\zeta_i}e^{\hat\zeta_j}) = 0$ means orientation of every *i*-th agent converge to the same value

**Each agents converge to same orientation**

Q.E.D.



## Proof of Time Delay Case



2. Proof of Time Delay Case



## Proof of Time Delay Case



***Proof***

Define the positive definite Lyapunov-Krasovskii function as the following function.

$$V_{delay} := \sum_{i=1}^{n}\gamma_i\phi(e^{\hat\zeta_i(t)}) + \sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\int_{t-T_{ji}}^{t}\phi(e^{\hat\zeta_i(\tau)})d\tau$$

The derivative of the positive definite Lyapunov-Krasovskii function is given as

$$\dot V_{delay} := \sum_{i=1}^{n}\gamma_i\dot\phi(e^{\hat\zeta_i(t)}) + \sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\frac{d}{dt}\left(\int_{t-T_{ji}}^{t}\phi(e^{\hat\zeta_i(\tau)})d\tau\right)$$

$$= \sum_{i=1}^{n}\gamma_i\left(\mathrm{sk}(e^{\hat\zeta_i(t)})^\vee\right)^T\omega_i^b + \sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\frac{d}{dt}\left(\int_{t-T_{ji}}^{t}\phi(e^{\hat\zeta_i(\tau)})d\tau\right)$$

$$= \sum_{i=1}^{n}\gamma_i\left(\mathrm{sk}(e^{\hat\zeta_i(t)})^\vee\right)^T\sum_{j\in N_i}k_{ij}\mathrm{sk}(e^{-\hat\zeta_i(t)}e^{\hat\zeta_j(t-T_{ij})})^\vee + \sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\frac{d}{dt}\left(\int_{t-T_{ji}}^{t}\phi(e^{\hat\zeta_i(\tau)})d\tau\right)$$

$$= \sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\left(\mathrm{sk}(e^{\hat\zeta_i(t)})^\vee\right)^T\mathrm{sk}(e^{-\hat\zeta_i(t)}e^{\hat\zeta_j(t-T_{ij})})^\vee + \sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\frac{d}{dt}\left(\int_{t-T_{ji}}^{t}\phi(e^{\hat\zeta_i(\tau)})d\tau\right)$$

## Proof of Time Delay Case

$$\left(\mathrm{sk}(e^{\hat{\xi}_i(t)})^\vee\right)^T \mathrm{sk}(e^{-\hat{\xi}_i(t)}e^{\hat{\xi}_j(t-T_{ij})})^\vee$$
$$\le -\phi(e^{\hat{\xi}_i(t)})+\phi(e^{\hat{\xi}_j(t-T_{ji})})-\frac{1}{2}\lambda_{\min}\left(e^{\hat{\xi}_i(t)}+e^{-\hat{\xi}_i(t)}\right)\phi(e^{-\hat{\xi}_i(t-T_{ji})}e^{\hat{\xi}_j(t)})$$

$$\frac{d}{dt}\left(\int_{t-T_{ji}}^{t}\phi(e^{\hat{\xi}_i(\tau)})d\tau\right)=\phi(e^{\hat{\xi}_i(t)})-\phi(e^{\hat{\xi}_i(t-T_{ji})})$$

$$=\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\left(-\phi(e^{\hat{\xi}_i(t)})+\phi(e^{\hat{\xi}_j(t-T_{ji})})-\frac{1}{2}\lambda_{\min}\left(e^{\hat{\xi}_i(t)}+e^{-\hat{\xi}_i(t)}\right)\phi(e^{-\hat{\xi}_i(t-T_{ji})}e^{\hat{\xi}_j(t)})\right)$$
$$+\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\left(\phi(e^{\hat{\xi}_i(t)})-\phi(e^{\hat{\xi}_i(t-T_{ji})})\right)$$

$$=\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\left(-\phi(e^{\hat{\xi}_i(t-T_{ji})})+\phi(e^{\hat{\xi}_j(t-T_{ji})})\right)=0$$

$$=-\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\lambda_{\min}\left(e^{\hat{\xi}_i(t)}+e^{-\hat{\xi}_i(t)}\right)\phi(e^{-\hat{\xi}_i(t-T_{ji})}e^{\hat{\xi}_j(t)})\le 0$$

Using LaSalle's Invariance Principle, we can prove convergence.

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## Proof of Theorem2'

3. Proof of Theorem2'

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## Proof of Theorem2'

Theorem 2 Consider the $n$ agents (1) together with the angular velocity given by (2). Under the assumptions (B), the following inequality is satisfied.

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\le\frac{\max_{i,j,i\ne j}\gamma_i\gamma_j}{\min_{i,j,i\ne j}\gamma_i\gamma_j}\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\frac{\varepsilon\lambda_{\min 2}(L_{wsym})}{n\max_{i,j,i\ne j}\gamma_i\gamma_j}\sum_{i=1}^{n}\gamma_i t}$$

where $\lambda_{\min 2}(L_{sym})$ : second minimum eigenvalue of $L_{sym}$

$$\left(e^{\hat{\xi}}\right)^T=\left[\left(e^{\hat{\xi}_1}\right)^T\ \cdots\ \left(e^{\hat{\xi}_n}\right)^T\right]\in R^{3n\times 3}\quad \varepsilon:=\min_{i,j,t}\lambda_{\min}\left(e^{-\hat{\xi}_i(t)}e^{\hat{\xi}_j(t)}+e^{-\hat{\xi}_j(t)}e^{\hat{\xi}_i(t)}\right)\quad \vec{1}:=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$

$$L_{wsym}:=\frac{1}{2}(L_{wsym}+L_{wsym}^T)\quad M=nI-\vec{1}\vec{1}^T$$ : graph Laplacian of complete graph

***Proof***

Define the potential function as the following function

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})$$

$$\min_{i,k,i\ne k}\gamma_i\gamma_k\sum_{i=1}^{n}\sum_{k=1}^{n}\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\le\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\le\max_{i,k,i\ne k}\gamma_i\gamma_k\sum_{i=1}^{n}\sum_{k=1}^{n}\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})$$

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## Proof of Theorem2'

The derivative of potential function is given as

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\dot{\phi}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})=\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^\vee\right)^T\left(-\omega_i^b+\omega_k^b\right)$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^\vee\right)^T\omega_k=-\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\left(\mathrm{sk}(e^{-\hat{\xi}_k}e^{\hat{\xi}_i})^\vee\right)^T\omega_k$$
$$=-\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^\vee\right)^T\omega_i$$

$$=-2\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^\vee\right)^T\omega_i\qquad \omega_i=\sum_{j\in N_i}k_{ij}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$$

$$=-2\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^\vee\right)^T\sum_{j\in N_i}\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$$

$$=-2\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\left(\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})^\vee\right)^T\mathrm{sk}(e^{-\hat{\xi}_i}e^{\hat{\xi}_j})^\vee$$

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## Proof of Theorem2'

$$=-2\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}-\frac{1}{8}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}-e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-e^{-\hat{\xi}_j}e^{\hat{\xi}_i}\right)\right)$$

$$=\frac{1}{4}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-e^{-\hat{\xi}_i}e^{\hat{\xi}_k}e^{-\hat{\xi}_j}e^{\hat{\xi}_i}-e^{-\hat{\xi}_k}e^{\hat{\xi}_i}e^{-\hat{\xi}_i}e^{\hat{\xi}_j}+e^{-\hat{\xi}_k}e^{\hat{\xi}_i}e^{-\hat{\xi}_j}e^{\hat{\xi}_i}\right)$$

$$=\frac{1}{4}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}2\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-e^{-\hat{\xi}_k}e^{\hat{\xi}_j}\right)$$

$$=\frac{1}{4}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}2\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-e^{-\hat{\xi}_k}e^{\hat{\xi}_j}\right)$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}e^{-\hat{\xi}_k}e^{\hat{\xi}_j}=\sum_{k=1}^{n}\gamma_k e^{-\hat{\xi}_k}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}e^{\hat{\xi}_j}$$
$$=\sum_{k=1}^{n}\gamma_k e^{-\hat{\xi}_k}\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}e^{\hat{\xi}_i}=\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}e^{-\hat{\xi}_k}e^{\hat{\xi}_i}$$

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## Proof of Theorem2'

$$=\frac{1}{4}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}2\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-I\right)\right)$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-I\right)\right)=\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(e^{-\hat{\xi}_k}e^{\hat{\xi}_j}-e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)=0$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\mathrm{tr}\left(\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}+e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_j}-I\right)\right)$$

$$=-\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j\in N_i}\gamma_i\gamma_k k_{ij}\lambda_{\min}\left(e^{-\hat{\xi}_i}e^{\hat{\xi}_k}+e^{-\hat{\xi}_k}e^{\hat{\xi}_i}\right)\mathrm{tr}\left(I-e^{-\hat{\xi}_i}e^{\hat{\xi}_j}\right)$$

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## Proof of Theorem2’



$$\leq -\frac{1}{2}\varepsilon\sum_{k=1}^{n}\gamma_k\sum_{i=1}^{n}\sum_{j\in N_i}\gamma_i k_{ij}\,\mathrm{tr}\left(I-e^{-\hat{\xi}_i}e^{\hat{\xi}_j}\right)$$

$$=\frac{1}{2}\varepsilon\sum_{k=1}^{n}\gamma_k\,\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(L_w\otimes I_3)e^{\hat{\xi}}\right)$$

$$=\frac{1}{2}\frac{\varepsilon}{n^2}\sum_{k=1}^{n}\gamma_k\,\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(ML_wM\otimes I_3)e^{\hat{\xi}}\right)$$

$$=\frac{1}{2}\frac{\varepsilon}{n^2}\sum_{k=1}^{n}\gamma_k\,\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(ML_{sym}M\otimes I_3)e^{\hat{\xi}}\right)$$

$$=\frac{1}{2}\frac{\varepsilon}{n^2}\sum_{k=1}^{n}\gamma_k\,\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(ML_{wsym}M\otimes I_3)e^{\hat{\xi}}\right)$$

$$=\frac{1}{2}\frac{\varepsilon}{n^2}\sum_{k=1}^{n}\gamma_k\,\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)(L_{wsym}\otimes I_3)(M\otimes I_3)e^{\hat{\xi}}\right)$$



## Proof of Theorem2’



$$\vec{1}^T(M\otimes I_3)e^{\hat{\xi}}x=0$$

$$\lambda_{\min2}(L_{wsym})x^T\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)(M\otimes I_3)e^{\hat{\xi}}x\leq x^T\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)(L_{wsym}\otimes I_3)(M\otimes I_3)e^{\hat{\xi}}x$$

$$\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)\left((\lambda_{\min2}(L_{wsym})I-L_{wsym})\otimes I_3\right)(M\otimes I_3)e^{\hat{\xi}}\leq 0$$

$$\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)\left((\lambda_{\min2}(L_{wsym})I-L_{wsym})\otimes I_3\right)(M\otimes I_3)e^{\hat{\xi}}\right)\leq 0$$

$$\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)(L_{wsym}\otimes I_3)(M\otimes I_3)e^{\hat{\xi}}\right)\leq -\lambda_{\min2}(L_{wsym})\,\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)(M\otimes I_3)e^{\hat{\xi}}\right)$$

$$\leq\frac{1}{2}\frac{\varepsilon}{n^2}\sum_{k=1}^{n}\gamma_k\lambda_{\min2}(L_{wsym})\,\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)(M\otimes I_3)e^{\hat{\xi}}\right)$$

$$=-\frac{1}{2}\frac{\varepsilon}{n}\sum_{k=1}^{n}\gamma_k(L_{sym})\,\mathrm{tr}\left(\left(e^{\hat{\xi}}\right)^T(M\otimes I_3)e^{\hat{\xi}}\right)$$



## Proof of Theorem2’



$$=-\frac{\varepsilon}{n}\sum_{k=1}^{n}\gamma_k(L_{sym})\sum_{i=1}^{n}\sum_{k=1}^{n}\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})$$

$$\leq-\frac{1}{\max\limits_{i,k,i\neq k}\gamma_i\gamma_k}\varepsilon\sum_{k=1}^{n}\gamma_k(L_{sym})\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})$$

$$\frac{d}{dt}\left(\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\right)\leq-\frac{1}{\max\limits_{i,k,i\neq k}\gamma_i\gamma_k}\varepsilon\sum_{k=1}^{n}\gamma_k(L_{sym})\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\leq\sum_{i=1}^{n}\sum_{k=1}^{n}\gamma_i\gamma_k\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})e^{-\frac{\varepsilon}{n}\frac{\lambda_{\min2}(L_{wsym})}{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}\sum\limits_{i=1}^{n}\gamma_i t}$$

$$\min_{i,j,i\neq j}\gamma_i\gamma_j\sum_{i=1}^{n}\sum_{k=1}^{n}\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\leq\max_{i,j,i\neq j}\gamma_i\gamma_j\sum_{i=1}^{n}\sum_{k=1}^{n}\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})e^{-\frac{\varepsilon}{n}\frac{\lambda_{\min2}(L_{wsym})}{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}\sum\limits_{i=1}^{n}\gamma_i t}$$



## Proof of Theorem2’



$$\sum_{i=1}^{n}\sum_{k=1}^{n}\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})\leq\frac{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}{\min\limits_{i,j,i\neq j}\gamma_i\gamma_j}\sum_{i=1}^{n}\sum_{k=1}^{n}\phi(e^{-\hat{\xi}_i}e^{\hat{\xi}_k})e^{-\frac{\varepsilon}{n}\frac{\lambda_{\min2}(L_{wsym})}{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}\sum\limits_{i=1}^{n}\gamma_i t}$$

$$\frac{1}{2}\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\leq\frac{1}{2}\frac{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}{\min\limits_{i,j,i\neq j}\gamma_i\gamma_j}\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\frac{\varepsilon}{n}\frac{\lambda_{\min2}(L_{wsym})}{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}\sum\limits_{i=1}^{n}\gamma_i t}$$

$$\mathrm{tr}\left(\left(e^{\hat{\xi}(t)}\right)^T(M\otimes I_3)e^{\hat{\xi}(t)}\right)\leq\frac{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}{\min\limits_{i,j,i\neq j}\gamma_i\gamma_j}\mathrm{tr}\left(\left(e^{\hat{\xi}(0)}\right)^T(M\otimes I_3)e^{\hat{\xi}(0)}\right)e^{-\frac{\varepsilon}{n}\frac{\lambda_{\min2}(L_{wsym})}{\max\limits_{i,j,i\neq j}\gamma_i\gamma_j}\sum\limits_{i=1}^{n}\gamma_i t}$$