On Scalar LQR Control with Communication Cost

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1 Introduction

Recently, the issue of control under communication constraints has been an active area of research. For example, studies on the stabilization of quantized feedback or system performance under various noisy channel models. At this paper the author would like approach the problem of controlling a system from an optimization perspective by adding a quadratic communication cost into the LQG problem formulation.

2 Problem Statement

Given a scalar discrete-time, stochastic linear system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, k \geq 0 \\ y_k &= Cx_k + v_k \end{aligned} \tag{2.1}$$

- $A \in \mathcal{R}$, $B \in \mathcal{R}$ are nonzero. $\{x_k \in \mathcal{R}\}$ is the state sequence、$\{u_k \in \mathcal{R}\}$ is the control sequence, and $\{y_k \in \mathcal{R}\}$ is the observed system output、$C \in \mathcal{R}_{++}$ is the observation gain. Disturbances $\{w_k \in \mathcal{R}\}$ and $\{w_k \in \mathcal{R}\}$are uncorrelated white gaussian noise process with zero mean and covariances $W \in \mathcal{R}_{++}$ and $V \in \mathcal{R}_{++}$. The initial condition, x_0 is zero mean , has covariance $\Pi_0 \in \mathcal{R}_{++}$, and is uncorrelated with $\{w_k\}$ and $\{v_k\}$.
- channel between the controller and the plant is an additive white Gaussian noise channel.
- controller has access to past and present observations, and also all past control signals. The control input is of the form

$$u_k = \mu(y_0, y_1, ..., y_k, u_0, u_1, ...u_{k-1})$$

- goal is to design the optimal control law, $\{u_k\}$ and the optimal measurement gain , C, to achieve the minimal average cost

$$J^* \triangleq \min_{C,\{u_k\}} \lim_{n\to\infty} E[\frac{1}{n}\sum_{k=0}^{n-1}(Q{x_k}^2 + R{u_k}^2 + SC^2)] \tag{2.2}$$

where $Q \in \mathcal{R}_{++}$, $R \in \mathcal{R}_{++}$, and $S \in \mathcal{R}++$.

3 LQG Problem

補題 3.1 *For a fixed $C \in \mathcal{R}_{++}$、The average cost for system (2.1) is*

$$J_{LQG} = \lim_{n\to\infty} E[\frac{1}{n}\sum_{k=0}^{n-1}(Q{x_k}^2 + R{u_k}^2)] \tag{3.1}$$

The optimal performance is:

$$J^*_{LQG} \triangleq \min_{\{u_k\}} J_{LQG} = PW + (A^2P - P + Q)\Gamma, \quad (3.2)$$

where P and Γ satisfy

$$P = A^2P - (APB)^2(B^2P + R)^{-1} + Q, \quad (3.3)$$
$$\Gamma = \Sigma - (\Sigma C)^2(C^2\Sigma + V)^{-1},$$
$$\Sigma = A^2\Sigma - (A\Sigma C)^2(C^2\Sigma + V)^{-1} + W \quad (3.4)$$

P and Σ is the positive roots of equations (3.3) and (3.4).

Some of the key ideas of LQG can be summarize:

- $u_k = L_k\hat{x}_k$, $\hat{x}$ is the optimal estimate of the current state. In the steady state, $u_k = L\hat{x}_k$, where $L = -(BPA)(B^2P + R)^{-1}$.
- optimal solution is obtained by solving the estimation problem and the control problem independently (The Separation).

Equations (3.3) and (3.4) are algebraic Ricatti equations (ARE) associated with the LQR and the linear estimation problem. We can state their solutions's conditions as a lemma.

補題 3.2 *If $Q > 0$ and $B \neq 0$, then there exists a $P > 0$ that is unique positive solution to the ARE (3.3). Similarly, if $V > 0$ and $C \neq 0$ then there exists a $\Sigma > 0$ that is unique positive solution to the ARE (3.4)*

4 Communication Constrained LQG

By Implementing the separation principle in the previous section, the optimal cost for the new formulation becomes

$$J^* \triangleq \min_C \{\min_{\{u_k\}} \lim_{n\to\infty} E[\frac{1}{n}\sum_{k=0}^{n-1}(Q{x_k}^2 + R{u_k}^2 + SC^2)]\} \quad (4.1)$$

and we have :

定理 4.1 *Given the linear system (2.1) and a fixed $C \in \mathcal{R}_{++}$,*

$$J^* \triangleq \min_{\{u_k\}} \lim_{n\to\infty} E[\frac{1}{n}\sum_{k=0}^{n-1}(Q{x_k}^2 + R{u_k}^2 + SC^2)] \quad (4.2)$$
$$= PW + C^2S + (A^2P - P + Q)\Gamma \quad (4.3)$$

where P and Γ are defined in (3.3) and (3.4)

The terms in (4.3) can be interperated as the cost associated with perfect information, PW; additional cost due to estimation error, $(A^2P - P + Q)\Gamma$; and cost due to communication, C^2S.
$From (3.3) we can write$

$A^2P - P + Q = \frac{(APB)^2}{B^2P+R}$
which implies

$$A^2P - P + Q > 0 \tag{4.4}$$

Also, From (3.4) we can write

$$\Sigma = \frac{A^2V\Sigma}{C^2\Sigma + V} + W$$

which implies

$$\begin{aligned} \frac{A^2V\Sigma}{C^2\Sigma+V} < \Sigma \quad &\Rightarrow \quad A^2\frac{V}{C^2\Sigma+V} < 1 & (4.5) \\ &\Rightarrow \quad A^2(\frac{V}{C^2\Sigma+V})^2 < 1 \\ &\Rightarrow \quad (AV)^2 < (C^2\Sigma+V)^2 & (4.6) \end{aligned}$$

To make more concise, Let‘s define a few more functions:

補題 4.1

$$h(x) \triangleq 3x^4 + 8x^3V + 2x^2V^2(A^2+3) - 4xV^3A^2(A^2-1) - V^4(A^2-1)^2) \tag{4.7}$$

then for $x > \max\{0, V(A^2-1)\}, h''(x) > 0$ *and* $h'(x) \geq 0$

証明 1

$$\begin{aligned} h'(x) &= 12x^3 + 24x^2V + 4xV^2(A^2+3) - 4V^3A^2(A^2-1) \\ h''(x) &= 36x^2 + 48xV + 4V^2(A^2+3) \end{aligned} \tag{4.8}$$

It is clear that for $x > 0$, $h''(x) > 0$. *If* $\mid A \mid < 1$, *then* $h'(0) = 4V^3A^2(1-A^2) > 0$. *If* $\mid A \mid \geq 1$, *then* $h'(V(A^2-1)) = 12V^3A^4(A^2-1) \geq 0$. *Thus,* $h'(x) \geq 0$ *for* $x > \max\{0, V(A^2-1)\}$.

Now, to examine the convexity of cost function 4.3, let

$$g(C) \triangleq \frac{2V(A^2P - P + Q)\Sigma^2}{((V + C^2\Sigma)^2) - (AV)^2)^3},$$

Then the second derivative of $J(C)$ with respect to C can be written as

$$J''(C) = 2S + g(C)h(C^2\Sigma) \tag{4.9}$$

定理 4.2 *When the system is unstable,* $\mid A \mid \geq 1$, *the cost* $J(C)$*is convex in* $C > 0$, *i.e.,* $J''(C) \geq 0$ *for* $C > 0$.

Unfortunately, when the system is stable, $\mid A \mid < 1$ so $h(0) < 0$. Since S can be chosen arbitrarily small, there may exist some small $C > 0$ such that $J''(C) < 0$. Therefore, J is not always convex in C.
However, we can prove something that's almost as good: quasiconvexity.

定義 4.1 *A function $\beta : \mathcal{R} \to \mathcal{R}$ is said to be quasiconvex if its domain and all its sublevel sets*

$$S_\alpha = \{x \in dom\beta \mid \beta(x) \leq \alpha\}$$

for $\alpha \in \mathcal{R}$ are convex.

J is quasiconvex in $C > 0$ when $\mid A \mid < 1$ by proving that $J(C)$ exhibits the following property.

補題 4.2 *Suppose $\beta : \mathcal{R} \to \mathcal{R}$ is twice differentiable. The function β is quasiconvex if and only if for all $x \in dom\beta$,*

$$\beta'(x) = 0 \to \beta''(x) \geq 0$$

To show J is quasiconvex, we first examine the relationship of Σ and $C^2\Sigma$.

補題 4.3 *Let*

$$\begin{aligned} f(C) &= C^2\Sigma \\ &\triangleq \frac{1}{2}(C^2W - V + A^2V) + \frac{1}{2}\sqrt{C^4W^2 + 2(1+A^2)C^2WV + (A^2-1)^2V^2} \end{aligned} \tag{4.11}$$

then for $C > 0$, we have
1) $\Sigma(C) > 0$ and $\Sigma^{'}(C) < 0$
2) $f(C) > 0$ and $f'(C) > 0$

補題 4.4 *For $\mid A \mid < 1$, there exists some positive C_1 where $J''(C) > 0$ for $C > C_1$ and $J''(C)$ is strictly increasing in C for $0 < C < C_1$.*

A direct result Lemma 4.4 is:

Corollary 1 *When $\mid A \mid < 1$, if J is not convex, then there exists a unique $0 < C_0 < C_1$ where*

- *$J''(C) < 0$ if $C < C_0$,*
- *$J''(C) = 0$ if $C = C_0$,*
- *$J''(C) > 0$ if $C > C_0$,*

定理 4.3 *When $\mid A \mid < 1$, if $J(C)$ is not convex in $C > 0$, it is quasiconvex in $C > 0$.*

証明 2 *Assume $J(C)$ is not convex in $C > 0$.*

$$J'(C) = \frac{2C(V^2S(1-A^2)) + C^4\Sigma^2S + V\Sigma(...))}{(1-A^2)V^2 + 2C^2V\Sigma + C^4\Sigma^2}$$

Clearly, $J'(0)=0$ so that means there exists a unique $C^* > C_0$ where $J'(C^*) = 0$. But $C^* > C_0$ means that $J''(C) > 0$ and quasiconvexity follows from Lemma 4.2.

Therefore the optimal solution to our communication constrained LQG is :
1) If $\mid A \mid \geq 1$, $J(C)$ is convex and the optimal C is $C > 0$ such that $J'(C) = 0$.
2) If $\mid A \mid < 1$ and $J(C)$ is convex, the optimal C is $C = 0$ with $J^*(C) = \dfrac{QW}{(1-A^2)}$.
3) If $\mid A \mid < 1$, and $J(C)$ is quasiconvex, the optimal C is $C > 0$ such that $J'(C) = 0$.

5 conclusion

An LQG problem with a quadratic communication cost has been formulated. It is showed that the optimization problem is quasiconvex in C and if the system is unstable, the problem is convex and it can be solved in a computationally efficient manner. The next step is to extend the problem to general vector systems.

参考文献

Chih-Kai Ko, Xiaojie Gao, Stephen Prajna and J. Schulman, “On Scalar LQG Control with Communication Cost,” Proceedings of the 44th IEEE CDC, 2005