Analysis and Design of Linear Control System –Part2-

Instructor: Prof. Masayuki Fujita

4th Lecture

11 Frequency Domain Design

11.5 Fundamental Limitations (pp.331 to 340)

Keyword : Right Half-Plane Poles and Zeros
Gain Crossover Frequency Inequality

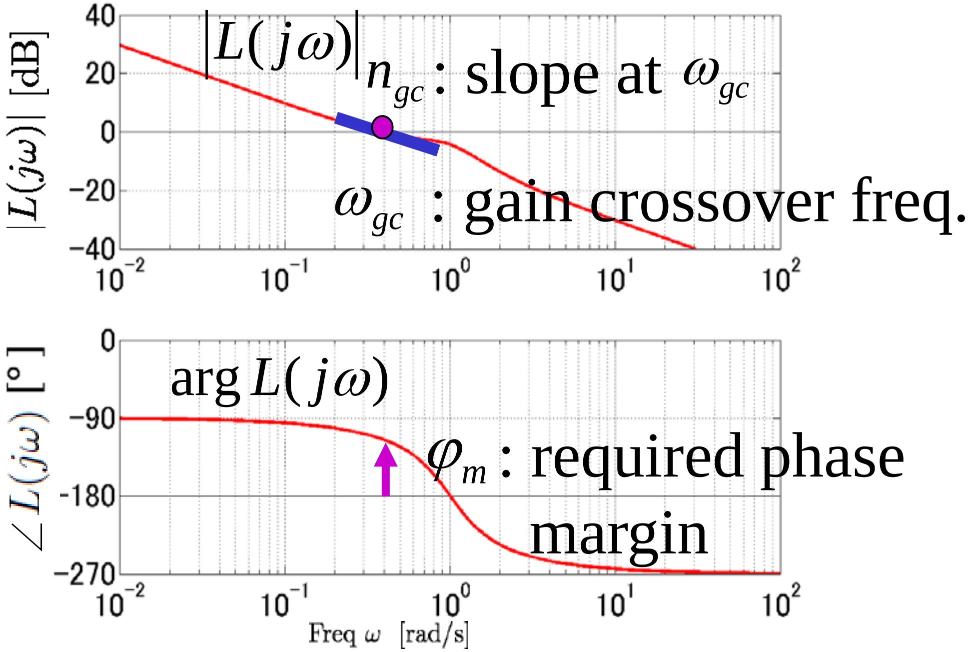
Recap. Gain Crossover Frequency Inequality

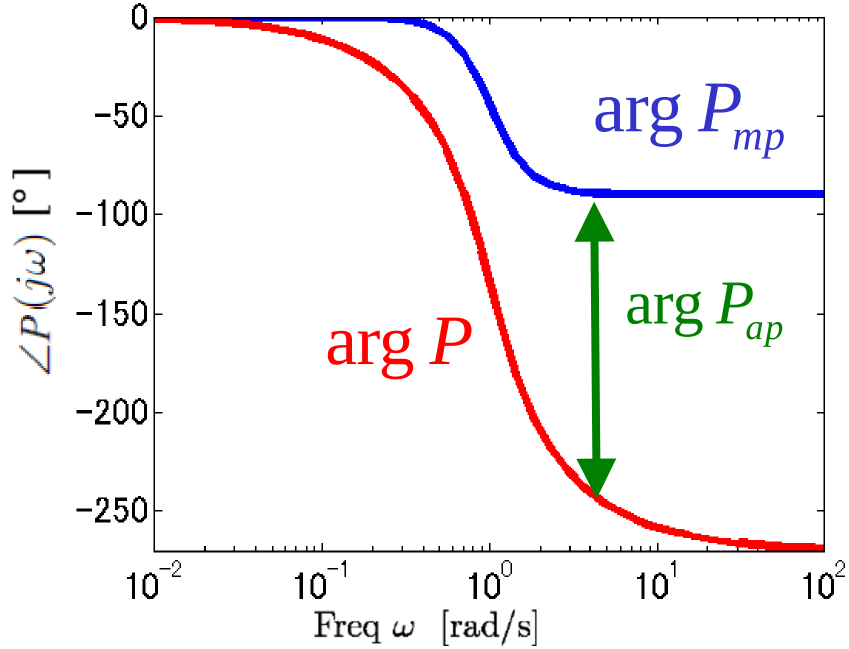
Factor the process transfer function as

$$P(s) = P_{mp}(s) P_{ap}(s)$$

P_{mp} : minimum phase part

P_{ap} : all-pass system





Gain Crossover Frequency Inequality

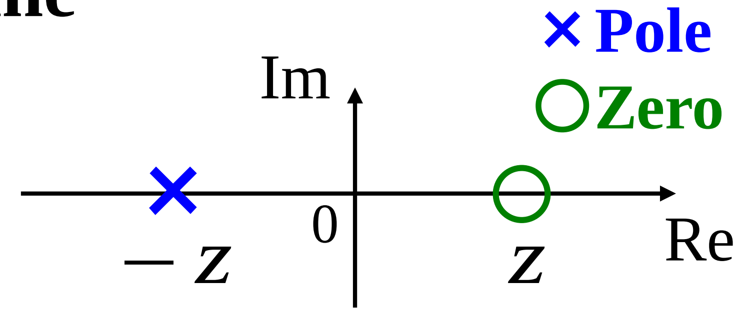
$$-\arg P_{ap}(j\omega_{gc}) \le \pi - \varphi_m + n_{gc}\frac{\pi}{2} =: \varphi_l \qquad (11.15)$$

⇒ allowable phase lag of P_{ap} at ω_{gc} : φ_l

[Ex. 11.7] Zero in the right half-plane

All-pass system with a RHP zero

$$P_{ap}(s) = \frac{z-s}{z+s} \quad z>0$$

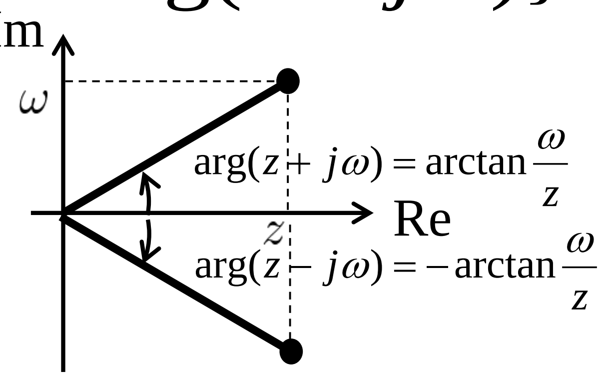


*RHP : right half-plane

Phase lag of the all-pass system

$$-\arg P_{ap}(j\omega) = -\{\arg(z-j\omega) - \arg(z+j\omega)\}$$

$$= 2\arctan\frac{\omega}{z}$$



gain crossover frequency inequality

$$-\arg P_{ap}(j\omega_{gc}) \le \pi - \varphi_m + n_{gc}\frac{\pi}{2} =: \varphi_l \quad (11.15)$$

Bound on the crossover frequency ω_{gc}

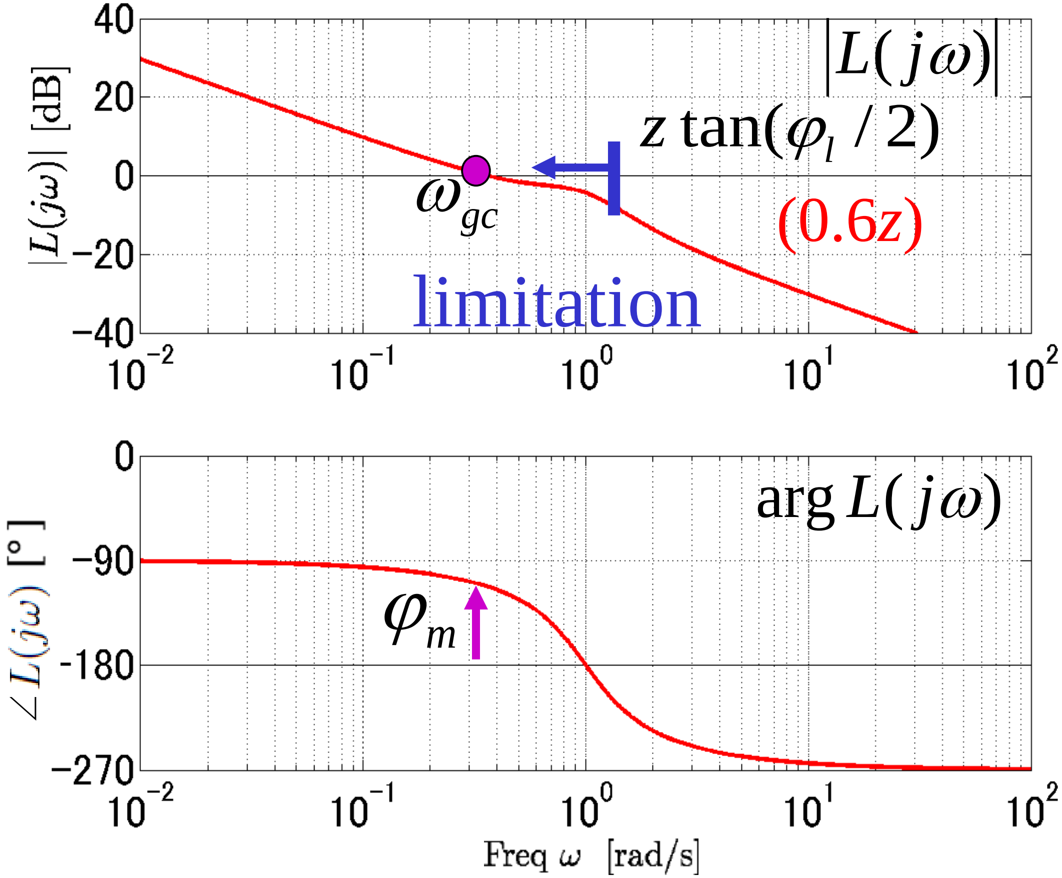
$$\omega_{gc} < z\tan(\varphi_l/2) \quad (11.16)$$

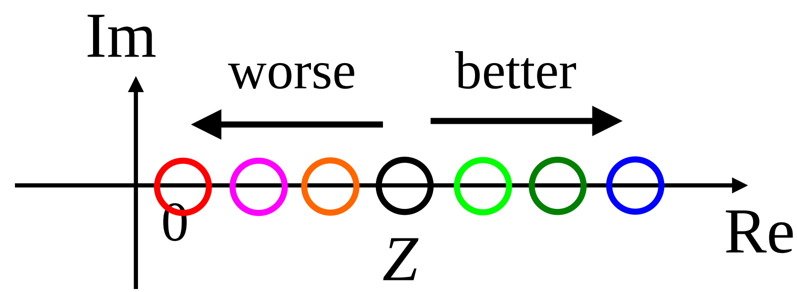
[Ex. 11.7] Zero in the right half-plane

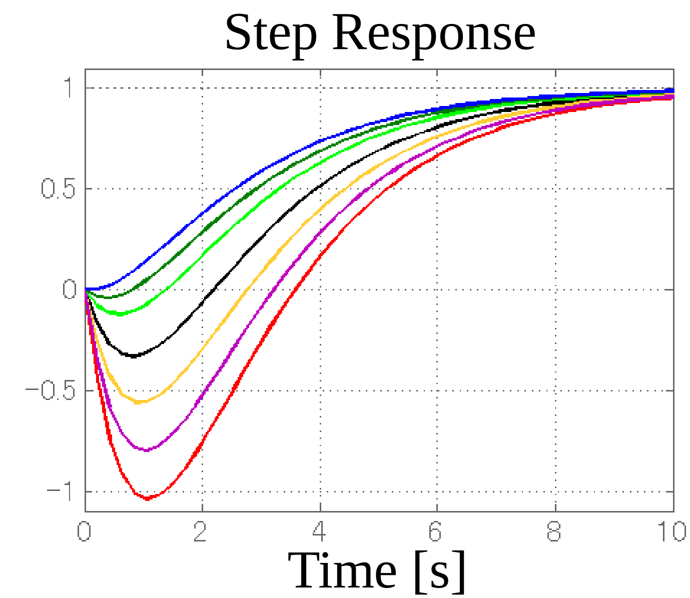
Bound on the crossover frequency ω_{gc}

$$\omega_{gc} < z\tan(\varphi_l / 2) \quad (11.16)$$

$$\varphi_l = 60^\circ \longrightarrow \omega_{gc} < 0.6z$$



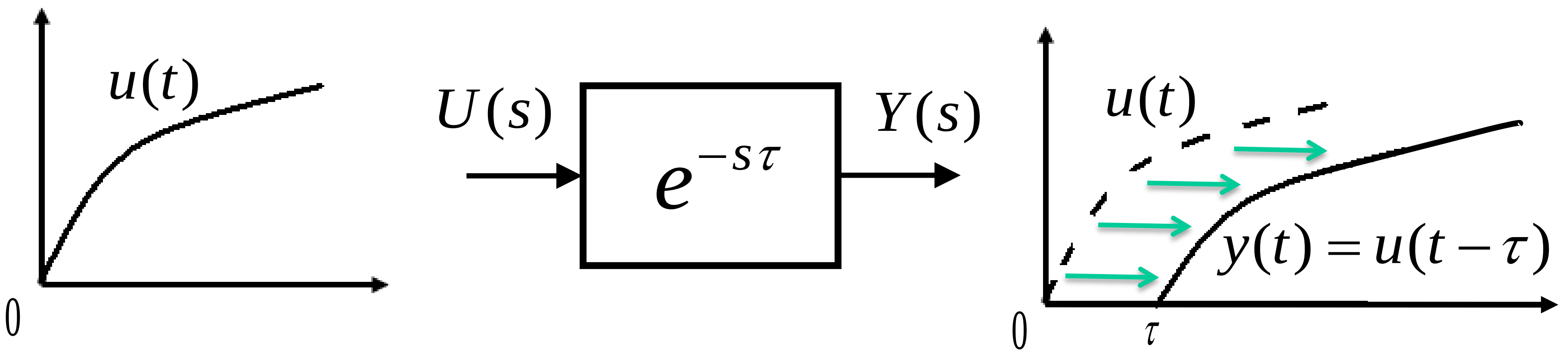




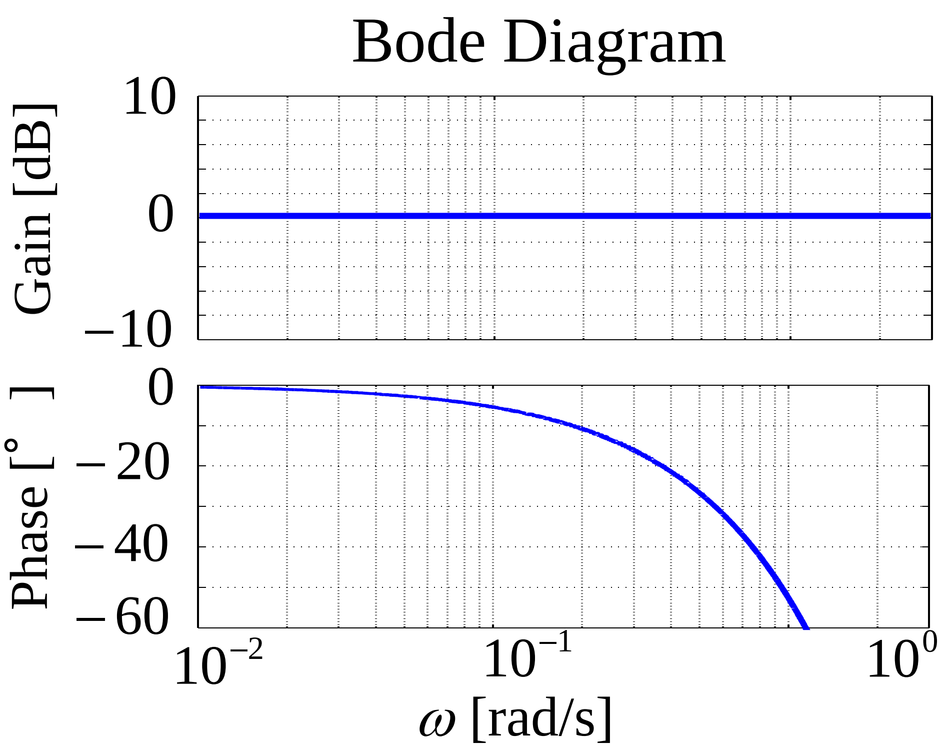
Slow RHP zeros (z small) → Tight restrictions

Fast RHP zeros (z large) → Loose restrictions

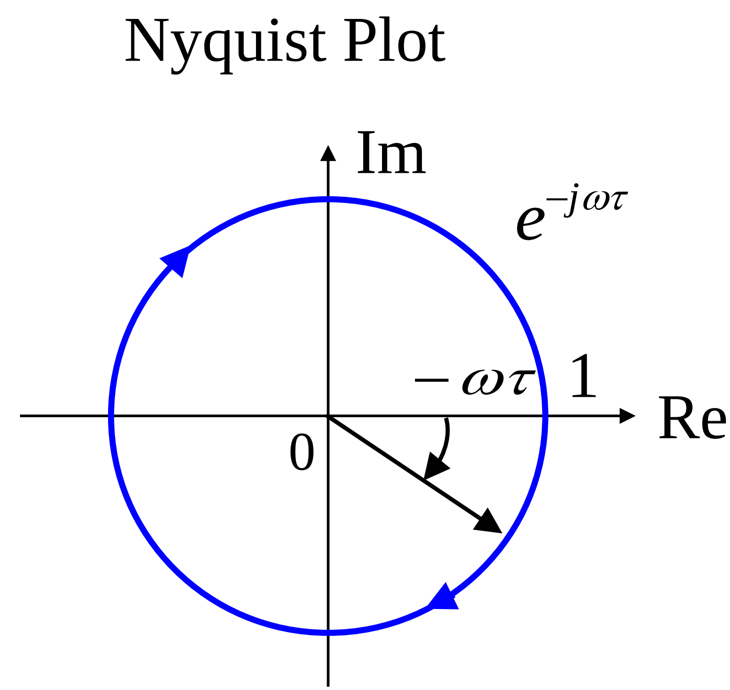
Time Delay



u(t)
0
U(s)
e^{-sτ}
Y(s)
u(t)
y(t) = u(t − τ)
0
τ

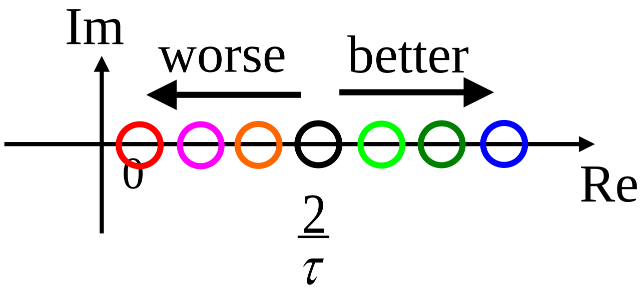


Bode Diagram
Gain [dB]
10
0
−10
Phase [°]
0
−20
−40
−60
10^{−2}
10^{−1}
10^{0}
ω [rad/s]



Nyquist Plot
Im
e^{−jωτ}
−ωτ
1
Re
0

Pade approximation



Time delay **Pade approximation**

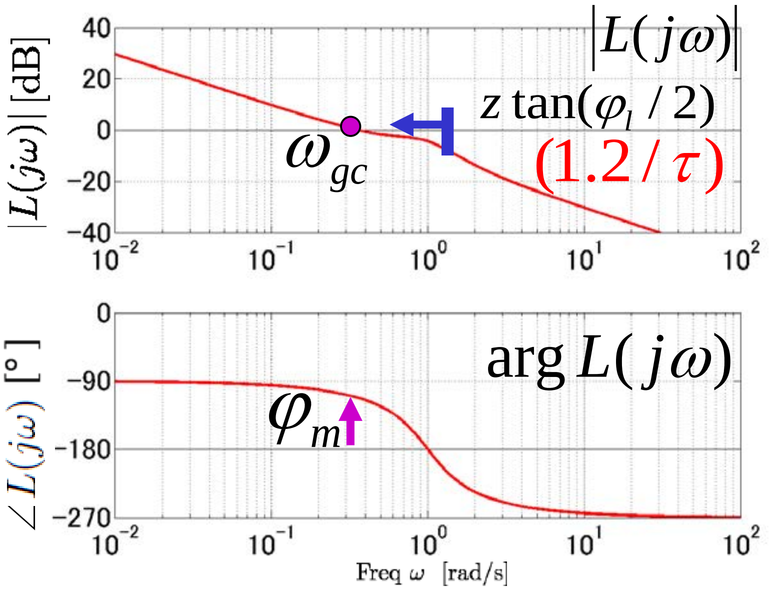
$$e^{-s\tau} \approx \frac{1-0.5s\tau}{1+0.5s\tau} = \frac{2/\tau - s}{2/\tau + s}$$

➡ Time delays also impose limitations similar to those given by zeros in the RHP.

A long time delay is equivalent to a slow RHP zero $z = 2/\tau$

$$\omega_{gc} < z\tan(\varphi_l/2) = \frac{2}{\tau}\tan(\varphi_l/2)$$

$$\varphi_l = 60^\circ \longrightarrow \omega_{gc} < \frac{1.2}{\tau}$$

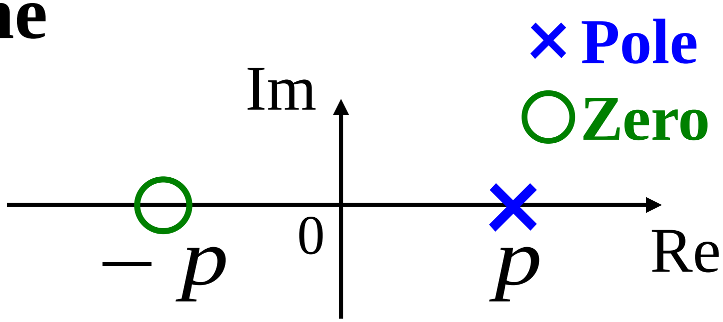


cf. sampling time vs. bandwidth

$$\frac{1}{40\tau} < f_c < \frac{1}{10\tau} \qquad f_c \text{ : bandwidth [Hz]}$$

[Ex. 11.8] Pole in the right half-plane

All-pass system with a RHP pole



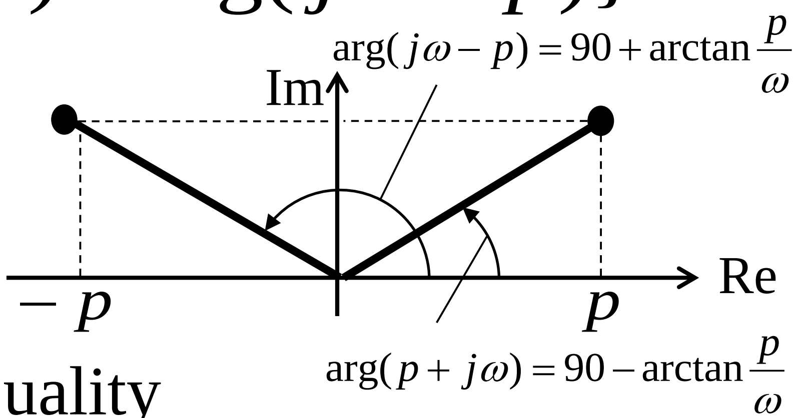
$$P_{ap}(s) = \frac{s+p}{s-p} \quad p > 0$$

*RHP : right half-plane

Phase lag of the all-pass system

$$-\arg P_{ap}(j\omega) = -\{\arg(p + j\omega) - \arg(j\omega - p)\}$$

$$= 2\arctan\frac{p}{\omega}$$



gain crossover frequency inequality

$$-\arg P_{ap}(j\omega_{gc}) \le \pi - \varphi_m + n_{gc}\frac{\pi}{2} =: \varphi_l \quad (11.15)$$

Bound on the crossover frequency ω_{gc}

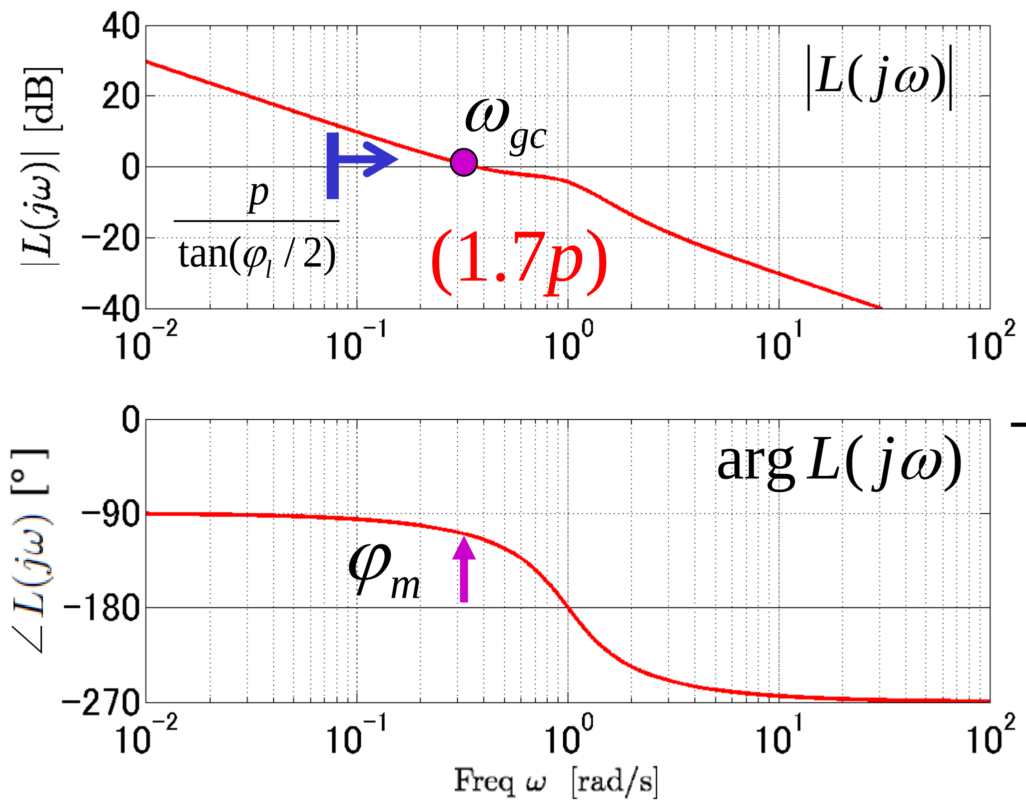
$$\omega_{gc} > \frac{p}{\tan(\varphi_l / 2)} \quad (11.17)$$

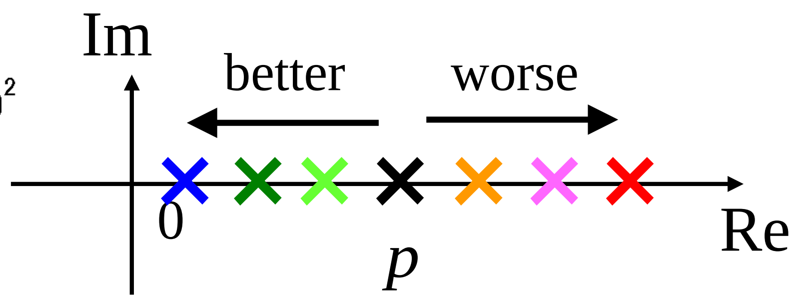
[Ex. 11.8] Pole in the right half-plane

Bound on the crossover frequency ω_{gc}

$$\omega_{gc} > \frac{p}{\tan(\varphi_l / 2)} \qquad (11.17)$$

$$\varphi_l = 60^\circ \longrightarrow \omega_{gc} > 1.7p$$

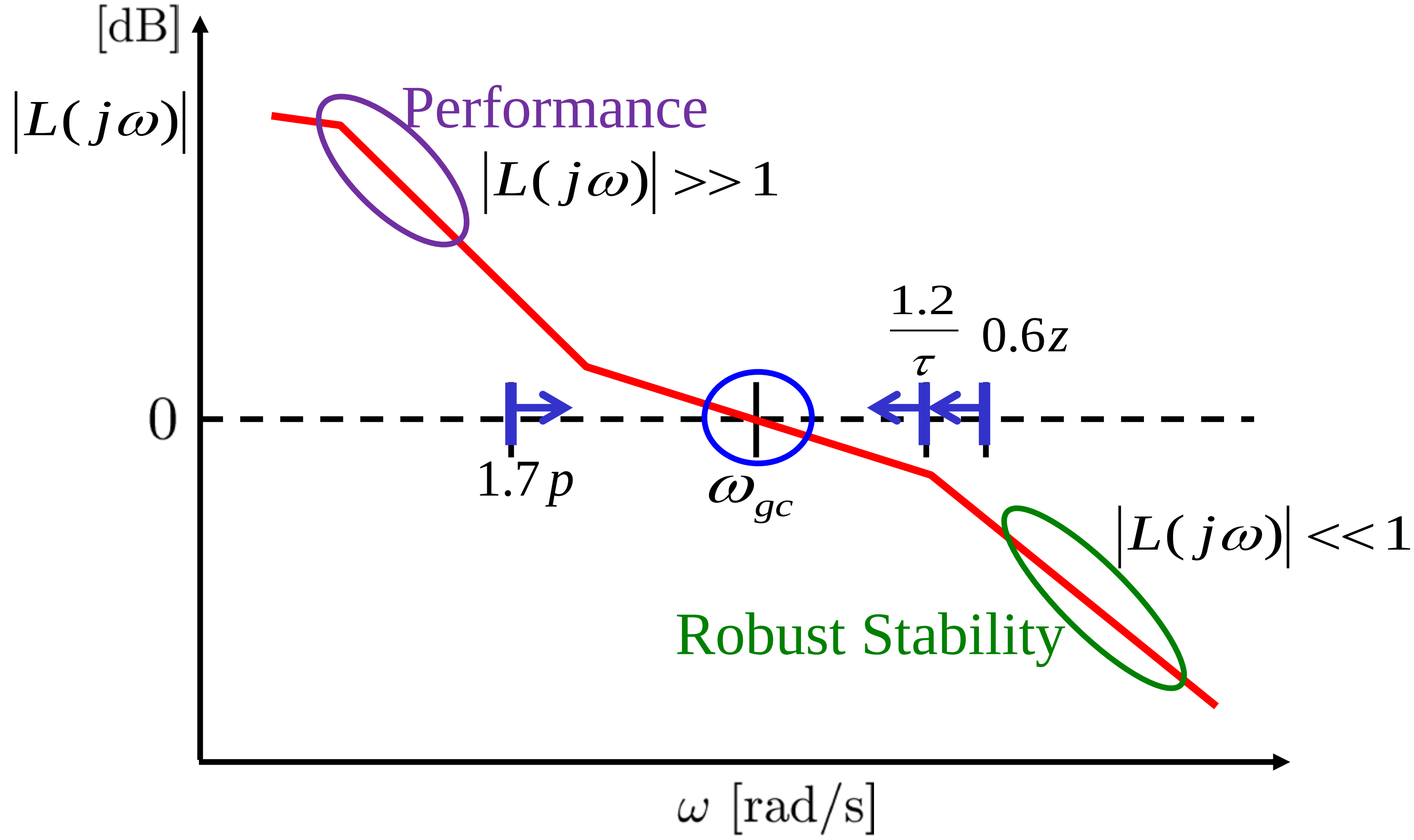




Fast RHP poles (p large) → Tight restrictions

Slow RHP poles (p small) → Loose restrictions

Loop Shaping



- RHP zero $\quad \omega_{gc} < z\tan(\varphi_l / 2)$
- Time Delay $\quad \omega_{gc} < z\tan(\varphi_l / 2) = \dfrac{2}{\tau}\tan(\varphi_l / 2)$
- RHP pole $\quad \omega_{gc} > \dfrac{p}{\tan(\varphi_l / 2)}$

Loop Shaping

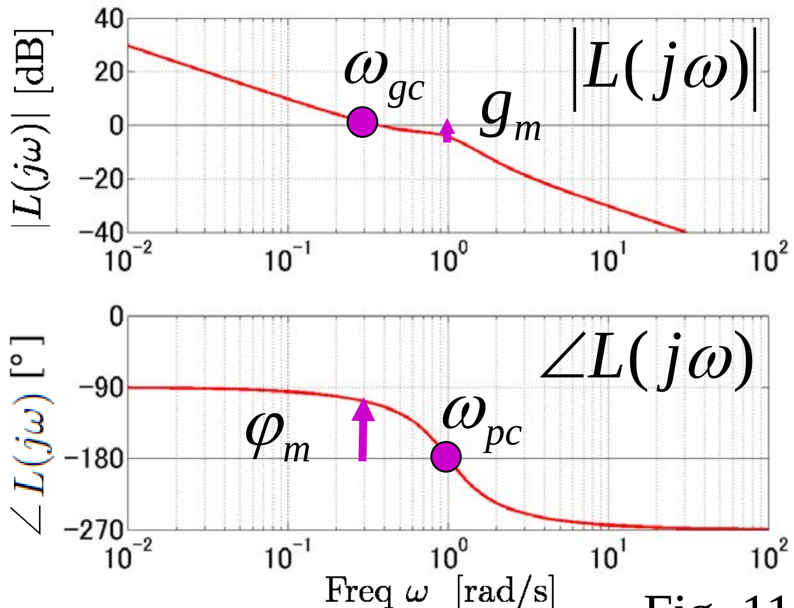
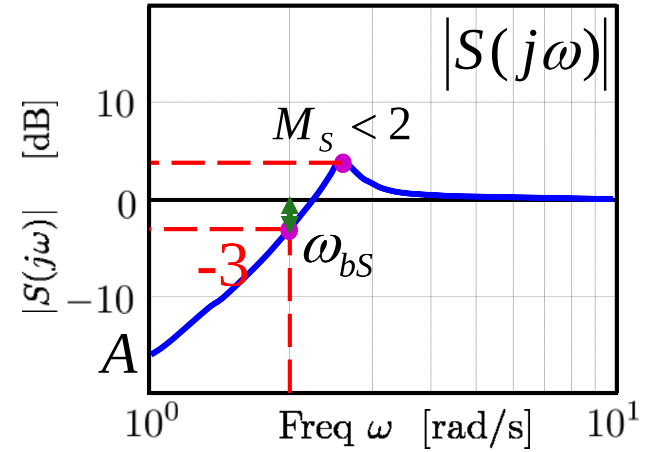
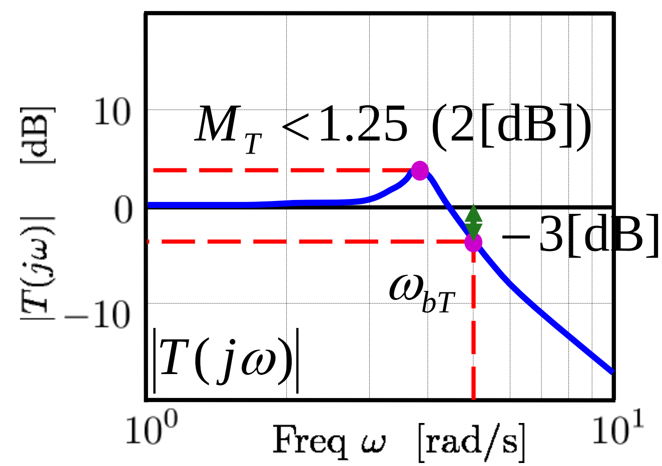


Fig. 11.8





- Gain Margin $g_m = 1/|L(j\omega_{pc})|$

  $(2-5)$

- Phase Margin $\varphi_m = \pi + \arg L(j\omega_{gc})$

  $(30° - 60°)$

- Stability Margin $s_m = 1/M_s$

  $(0.5-0.8)$

$M_S < 2 \qquad M_T < 1.25$

$$g_m \geq \frac{M_S}{M_S - 1} \qquad \varphi_m \geq 2\arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S}$$

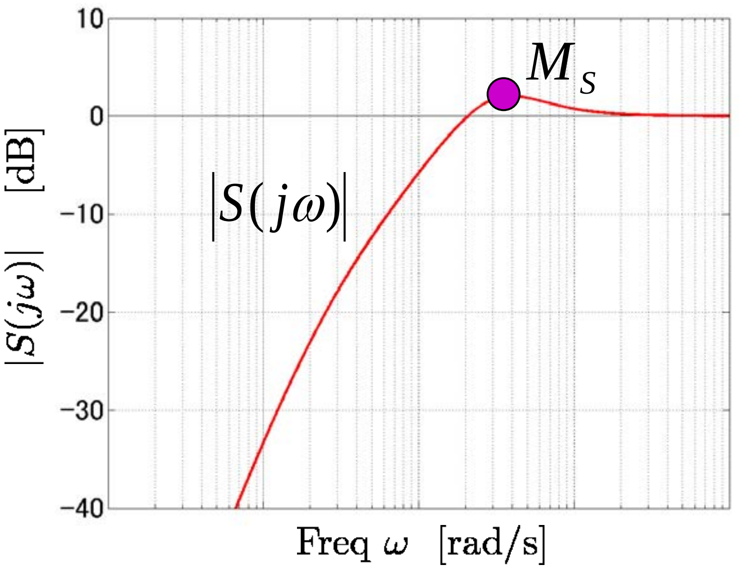
$$g_m \geq 1 + \frac{1}{M_T} \qquad \varphi_m \geq 2\arcsin\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T}$$

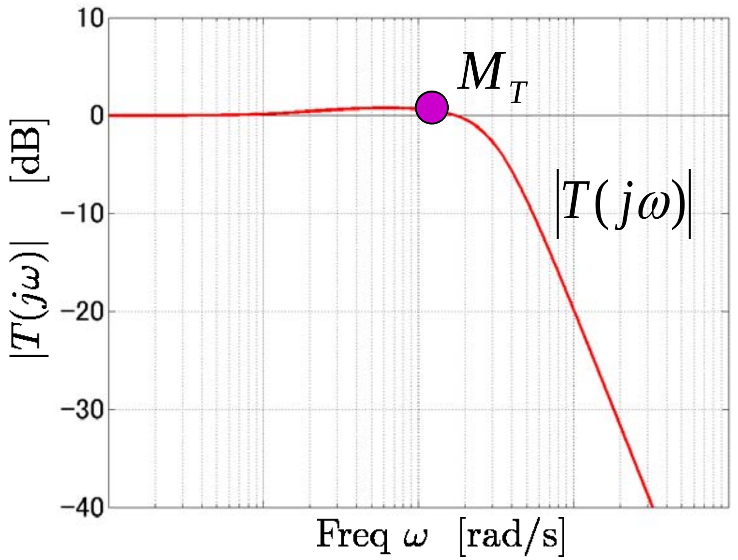
Right Half-Plane Poles and Zeros and Time Delays

For systems with a RHP pole p and RHP zero z (or a time delay τ), any stabilizing controller gives sensitivity functions with the property

$$M_S = \sup_{\omega} |S(j\omega)| \geq \frac{p+z}{|p-z|} \qquad M_T = \sup_{\omega} |T(j\omega)| \geq e^{p\tau} \qquad (11.18)$$

RHP pole and zero and time delay significantly limit the achievable performance of a system





Right Half-Plane Poles and Zeros and Time Delays

If RHP pole and zero are equal ($p = z$), there will be an unstable subsystem that is neither reachable nor observable, and the system cannot be stabilized

The zeros and the pole must be sufficiently far apart

Ex.)

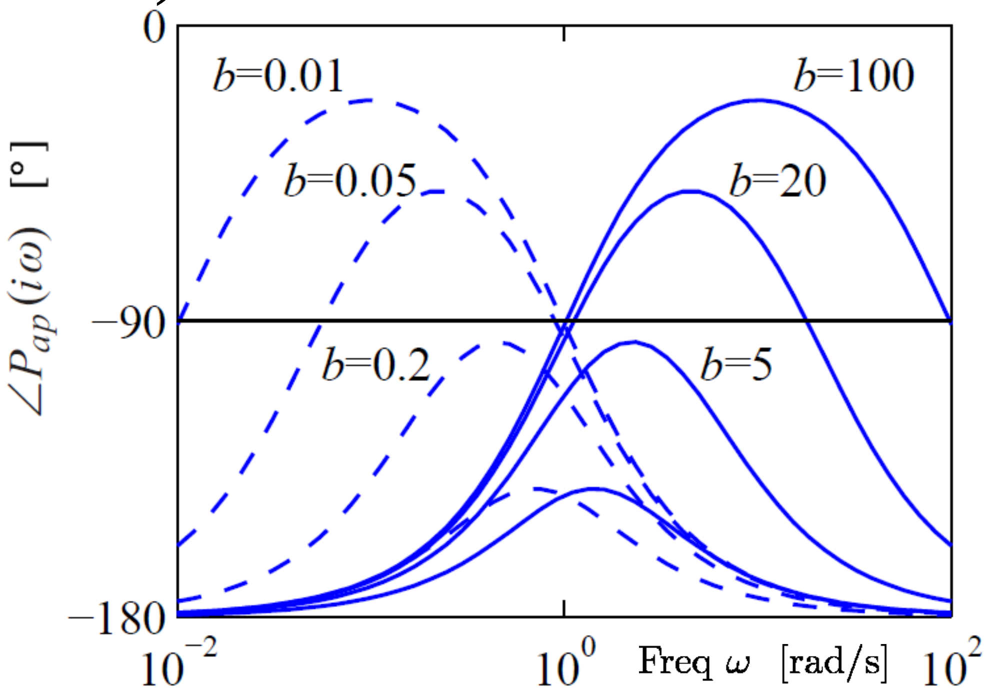


Fig. 11.13 (a) RHP pole/zero pair

all-pass system

$$P_{ap}(s) = \frac{b-s}{s-1} \qquad * \; p = 1, \; z = b$$

allowable phase lag of P_{ap} at ω_{gc}

$$\varphi_l = 90^\circ$$

$$-\arg P_{ap}(j\omega_{gc}) \le \varphi_l \qquad (11.15)$$

$$\frac{z}{p} < \frac{1}{6} \quad \text{or} \quad 6 < \frac{z}{p}$$

Right Half-Plane Poles and Zeros and Time Delays

The product of RHP pole and time delay must be sufficiently small

Ex.)

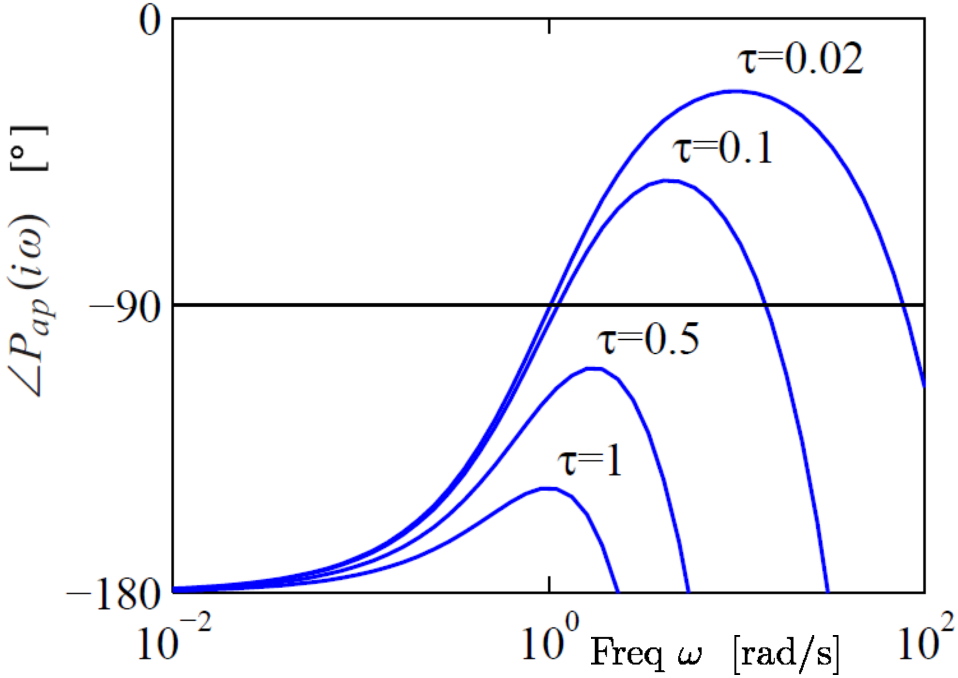


Fig. 11.13 (b) RHP pole and time delay

all-pass system

$$P_{ap}(s) = \frac{e^{-s\tau}}{s-1}$$

allowable phase lag of P_{ap} at ω_{gc}

$$\varphi_l = 90^\circ$$

$$-\arg P_{ap}(j\omega_{gc}) \le \varphi_l \quad (11.15)$$

$$p\tau < 0.3$$

[Ex. 11.9] Balance system (§ 6.3)

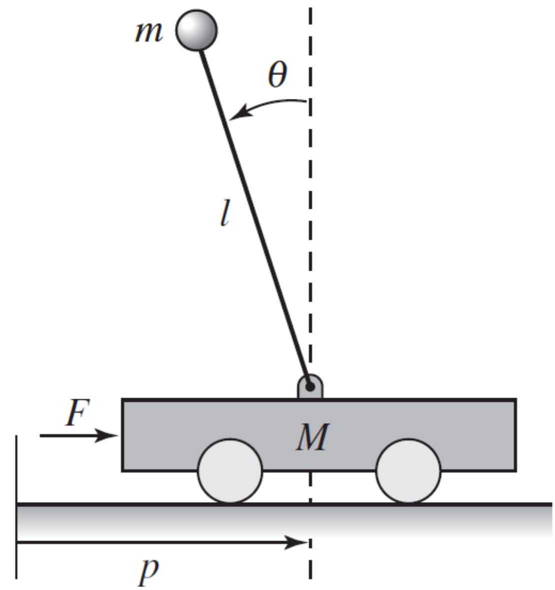


Fig. 6.2 (b)

Equations of motion

$$(M+m)\ddot{p} - ml\cos\theta\,\ddot{\theta} = -c\dot{p} - ml\sin\theta\,\dot{\theta}^2 + F$$

$$(J+ml^2)\ddot{\theta} - ml\cos\theta\,\ddot{p} = -\gamma\dot{\theta} + mgl\sin\theta \qquad (6.4)$$

Transfer functions

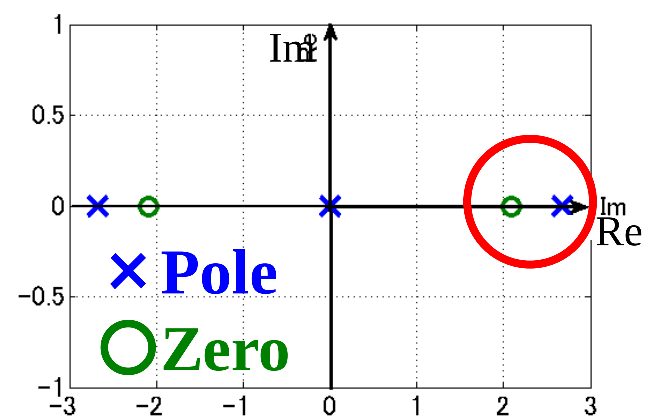
from F to θ $\quad H_{\theta F} = \dfrac{ml}{-(M_t J_t - m^2 l^2)s^2 + mglM_t}$

from F to p $\quad H_{pF} = \dfrac{-J_t s^2 + mgl}{s^2(-(M_t J_t - m^2 l^2)s^2 + mglM_t)}$

$$J_t = J + ml^2$$

H_{pF} : RHP pole $p = 2.68$

RHP zero $z = 2.09$



[Ex. 11.9] Balance system (§ 6.3)

$$\omega_{gc} > \frac{p}{\tan(\varphi_l / 2)}$$

RHP pole $p = 2.68$

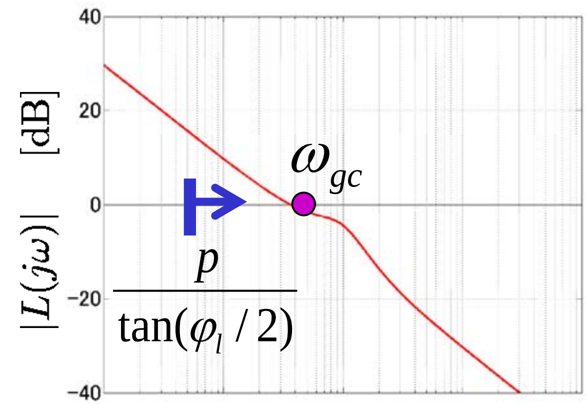
RHP zero $z = 2.09$

RHP zero can be eliminated

→ The gain crossover frequency inequality (11.15) is based just on the RHP pole

$$\varphi_l = 45^\circ \longrightarrow \omega_{gc} > 6.47 \ (\omega_{gc} > 2.4p)$$

$$\varphi_l = 60^\circ \longrightarrow \omega_{gc} > 4.56 \ (\omega_{gc} > 1.7p)$$



If the actuators have sufficiently high bandwidth, e.g. a factor of 10 above ω_{gc} or roughly 10 Hz, then we can provide robust tracking up to this frequency

$$M_S = \sup_{\omega} |S(j\omega)| \ge \frac{p+z}{|p-z|} \longrightarrow \sup_{\omega} |S(j\omega)| \ge 8$$

Ideally… $|S(j\omega)| < 2$ ⟶ difficult to control robustly

[Ex. 11.11] X-29 aircraft

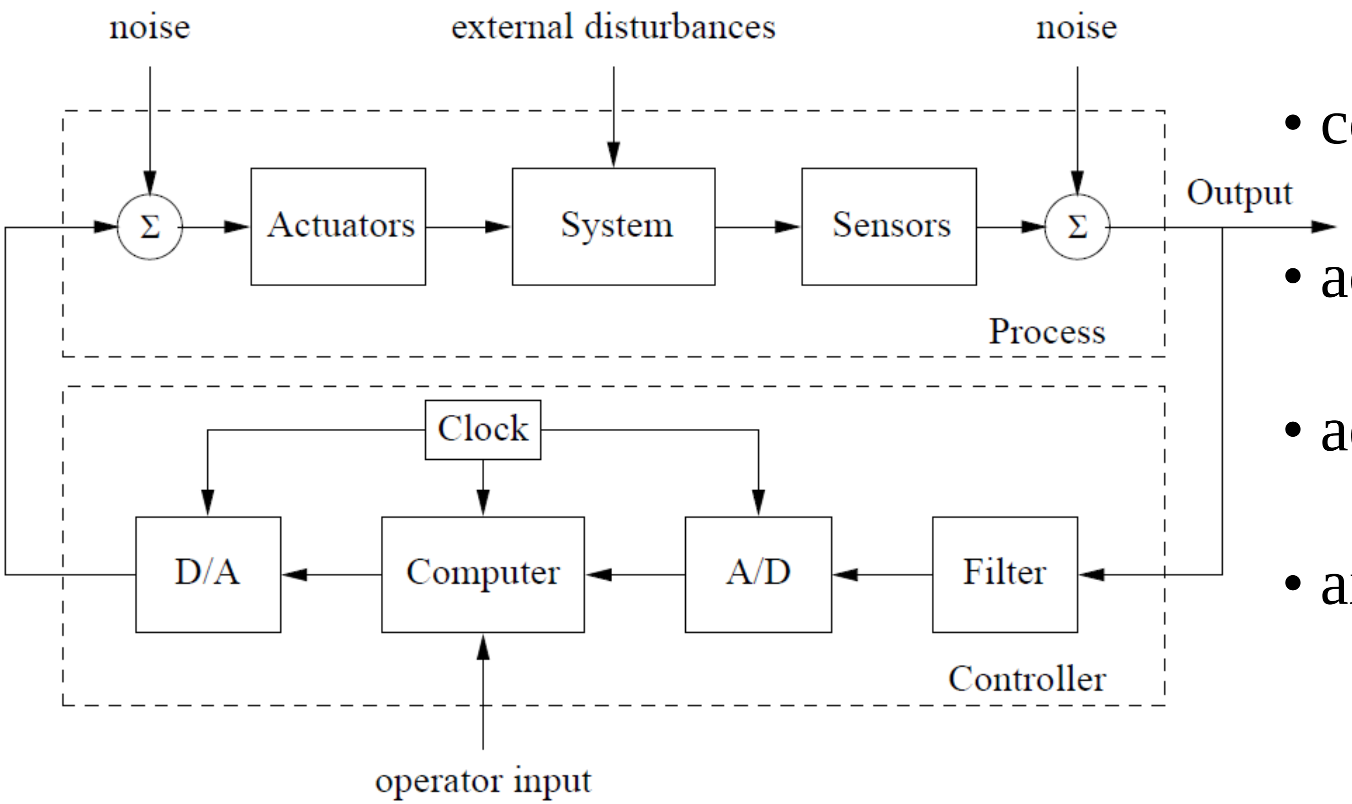


Fig. 1.3 Components of computer-controlled system

available bandwidth

- sensors
  : 120 rad/s
- control processors
  : 30-40 rad/s
- actuators
  : 70 rad/s
- aerodynamics
  : 100 rad/s
- airframe
  : 40 rad/s

Real physical systems have a multitude of limitations on available bandwidth

[Ex. 11.11] X-29 aircraft

X-29 longitudinal dynamics

- available bandwidth of the actuators that stabilize the pitch : $\omega_a = 40$ [rad/s]
- desired bandwidth of the pitch control loop : $\omega_1 = 3$ [rad/s]

Assume that the sensitivity function $S(s)$ is given

$$|S(j\omega)| = \frac{\omega M_s}{\omega_1} \quad (\omega \le \omega_1) \qquad |S(j\omega)| = M_s \quad (\omega_1 \le \omega \le \omega_a)$$

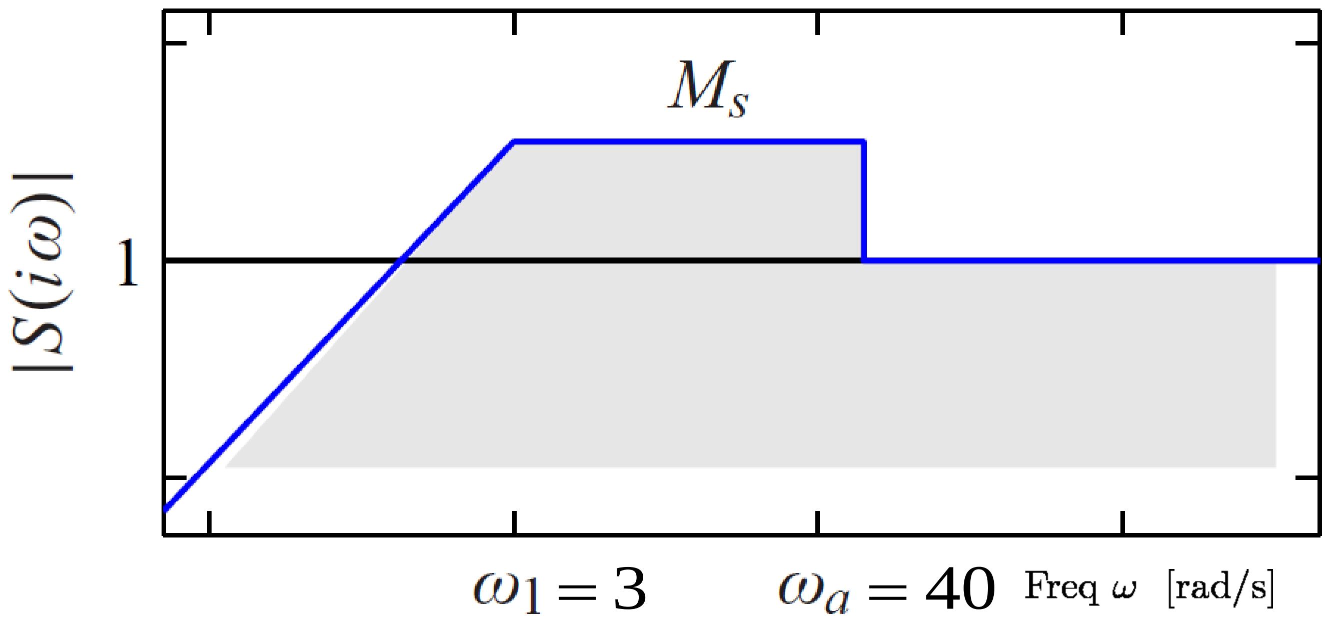


Fig 11.15 (b) Sensitivity analysis

[Ex. 11.11] X-29 aircraft

Assume $\left|L(s)\right| \le \delta / \omega^2 \quad \forall \omega \ge \omega_a$

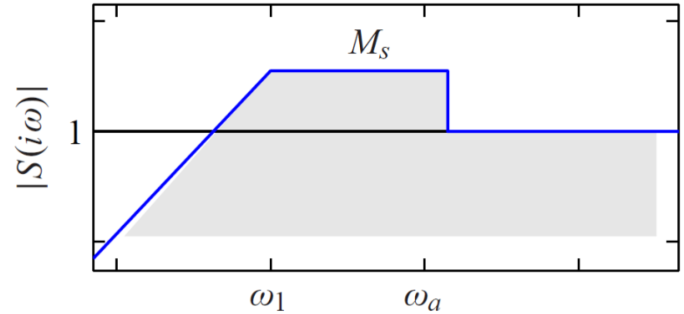


Fig 11.5 (b)

Bode's integral

$$\int_0^\infty \log\left|S(j\omega)\right| d\omega = \int_0^{\omega_a} \log\left|S(j\omega)\right| d\omega$$

$$= \int_0^{\omega_1} \log \frac{\omega M_s}{\omega_1} d\omega + (\omega_a - \omega_1) \log M_s$$

$$= \pi p$$

$p = 6$

$\omega_1 = 3$ [rad/s]

$\omega_a = 40$ [rad/s]

$$M_s = e^{(\pi p + \omega_1)/\omega_a} = 1.75 \quad (< 2)$$

$$M_S \approx \left|S(j\omega_{gc})\right| = \frac{1}{2\sin(\varphi_m / 2)}$$

maximum achievable phase margin : 35°

[Ex. 11.11] X-29 aircraft

X-29 aircraft

- maximum achievable phase margin : 35°

Boundaries for standard flight control specifications

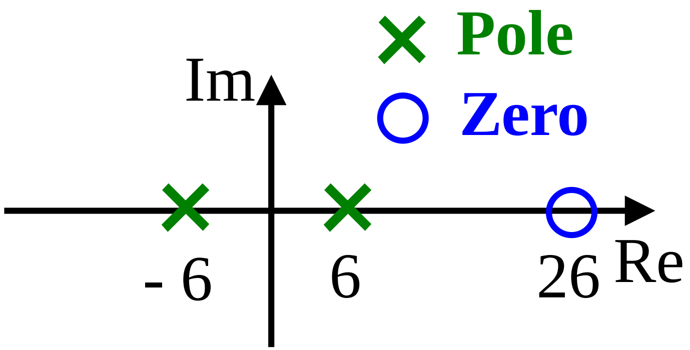
- phase margin : 45°

X-29 is difficult to control.

X-29 longitudinal dynamics

- poles : $p = \pm 6$
- zeros : $z = 26$

$$\frac{z}{p} = \frac{26}{6} \approx 4.3$$



$z/p > 6$ or $z/p < 1/6$

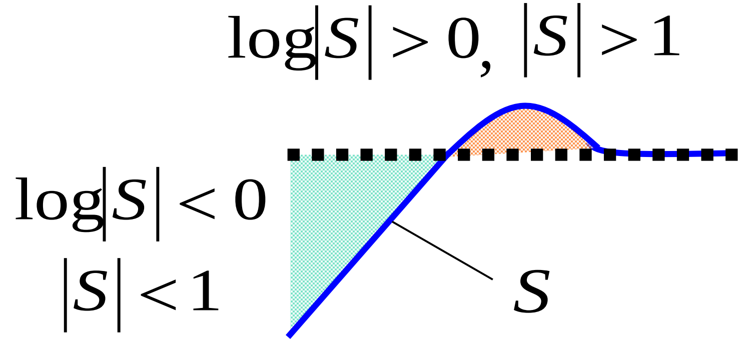
Desirable Condition

It is difficult to achieve the specifications, ***no matter how the controller is designed.***

Bode's Integral Formula (§ 11.5)

$$\int_0^\infty \log|S(j\omega)|\,d\omega = \pi \sum p_k \quad (11.19)$$

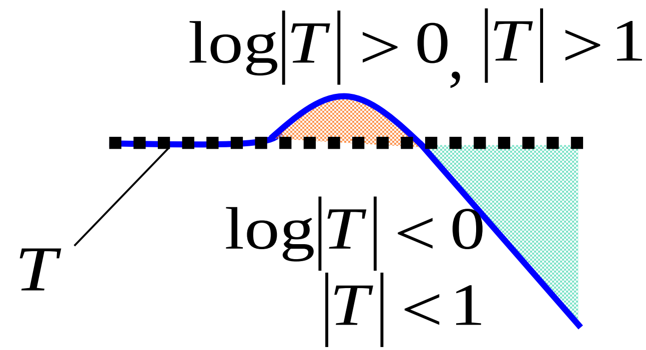
p_k : right half-plane poles



Waterbed Effect

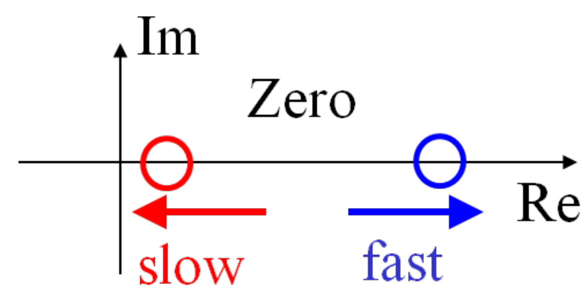
$$\int_0^\infty \frac{\log|T(j\omega)|}{\omega^2}\,d\omega = \pi \sum \frac{1}{z_i} \quad (11.20)$$

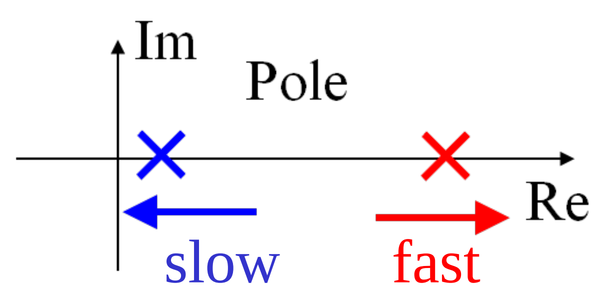
z_i : right half-plane zeros



	Better	Worse
RHP zeros	Fast (big)	Slow (small)
RHP poles	Slow (small)	Fast (big)

$$S + T = 1$$





4th Lecture

11 Frequency Domain Design

11.5 Fundamental Limitations (pp.331 to 340)

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