Analysis and Design of Linear Control System –Part2-

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3rd Lecture

11 Frequency Domain Design

11.4 Feedback Design via Loop Shaping (pp.326 to 331)

(9.4 Bode's Relations and Minimum Phase Systems) (pp.283 to 285)

Keyword : Loop Shaping
Bode's Relations

11.5 Fundamental Limitations (pp.331 to 340)

Keyword : Right Half-Plane Poles and Zeros
Gain Crossover Frequency Inequality

11.4 Feedback Design via Loop Shaping

Loop transfer function

$$L(s) = P(s)C(s)$$

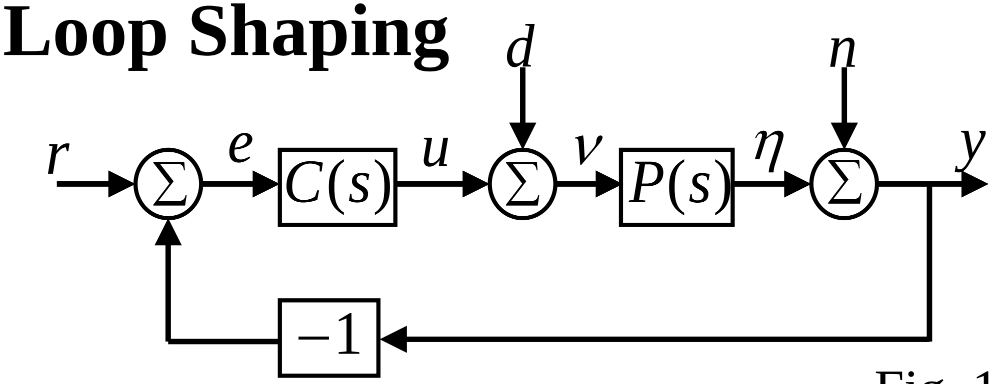


Fig. 11.1

Sensitivity

$$S(s) = \frac{1}{1+L}$$

- Load disturbance

$$G_{yd} = PS \quad (11.8) \qquad \frac{dG_{yd}}{G_{yd}} = S\frac{dP}{P} \quad (12.11)$$

- Tracking

$$\frac{dG_{yr}}{G_{yr}} = S\frac{dP}{P} \quad (12.15)$$

Complementary Sensitivity

$$T(s) = \frac{L}{1+L}$$

- Robust stability

$$|\delta| < 1/|T| \quad (12.6)$$

- Measurement noise

$$\frac{dG_{un}}{G_{un}} = T\frac{dP}{P} \quad (12.13)$$

Trade off $S + T = \frac{1}{1+PC} + \frac{PC}{1+PC} = 1$

11.4 Feedback Design via Loop Shaping

Loop transfer function

$$L(s) = P(s)C(s)$$

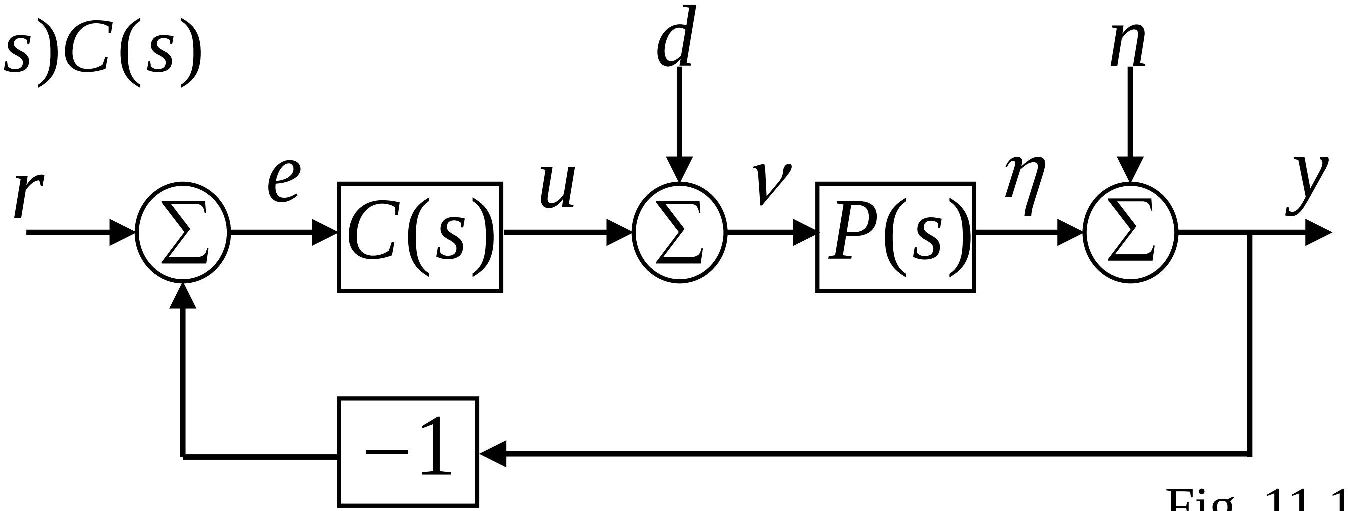


Fig. 11.1

Loop shaping

Choosing a compensator $C(j\omega)$ that gives a loop transfer function $L(j\omega)$ with a desired shape

➡ improve not only stability (Nyquist) but also performance and robustness

Loop Shaping

At low frequencies

$$|L(j\omega)| > 101 \longrightarrow |S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| < \frac{1}{100} \quad \left(\left|\frac{1}{1+L}\right| \le \left|\frac{1}{1-|L|}\right| < \frac{1}{100}\right)$$

Loop Gain

Feedback Performance

- Load disturbances will be attenuated by a factor of 100

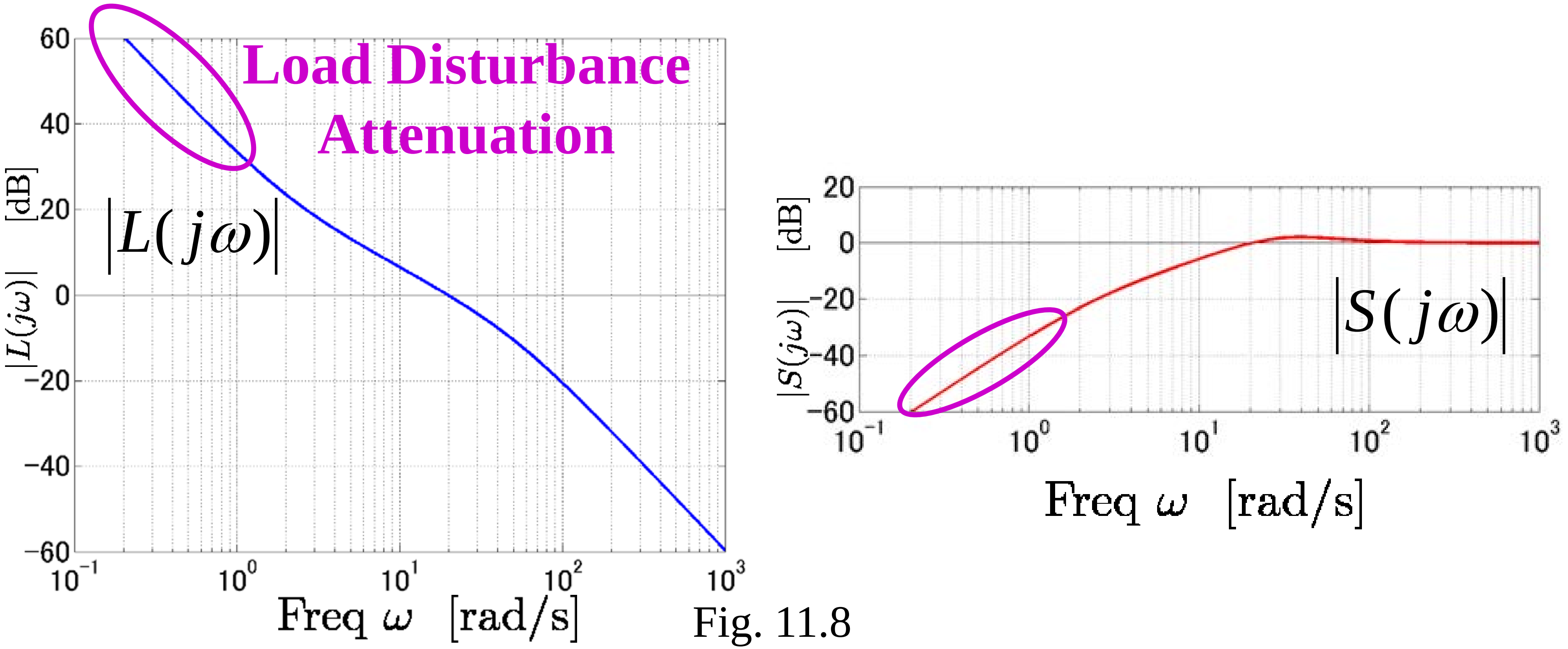


Fig. 11.8

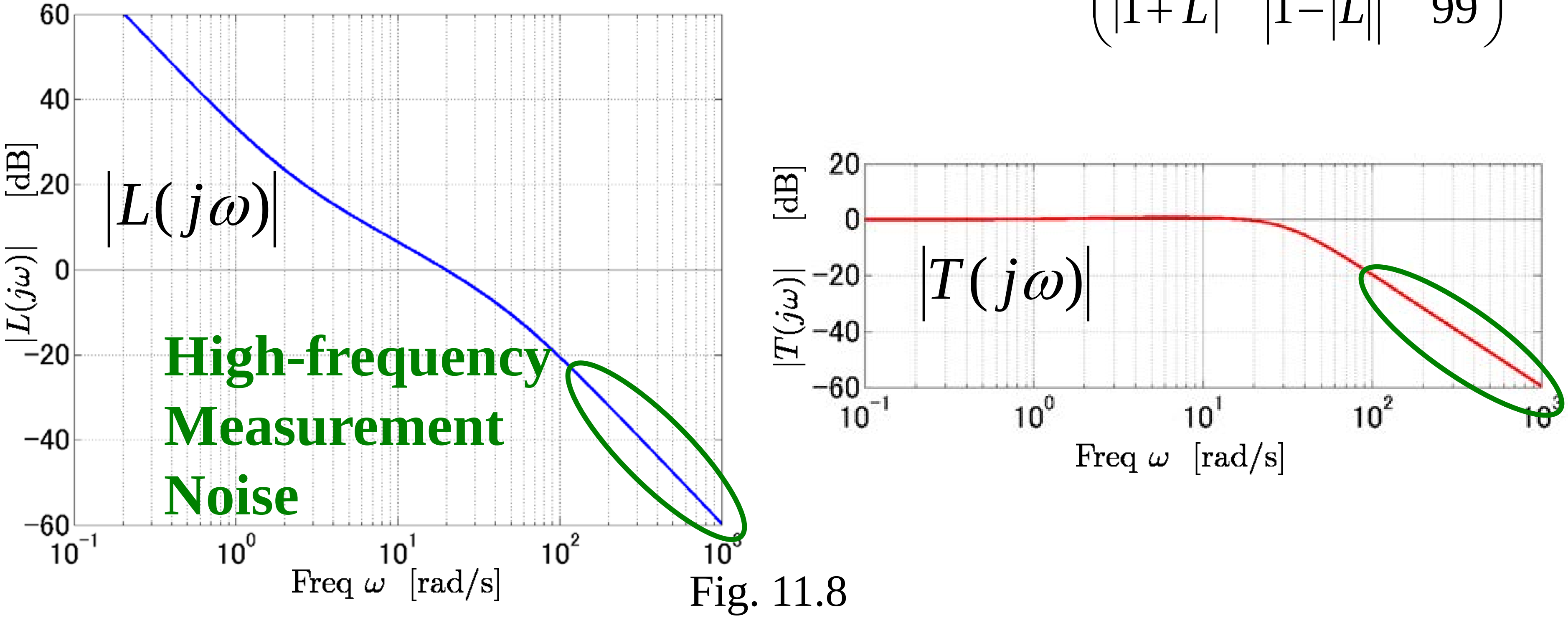
(a) Frequency response ($|L(j\omega)|$) (b) Frequency esponse ($|S(j\omega)|$)

Loop Shaping

At high frequencies

$$\underset{\text{Loop Gain}}{|L(j\omega)|} < 0.01 \longrightarrow |T(j\omega)| = \left|\frac{L(j\omega)}{1+L(j\omega)}\right| < \frac{1}{99} \approx 0.01$$

$$\left(\left|\frac{L}{1+L}\right| \le \left|\frac{L}{1-|L|}\right| < \frac{1}{99}\right)$$



$|L(j\omega)|$

High-frequency Measurement Noise

$|T(j\omega)|$

Fig. 11.8

(a) Frequency response ($|L(j\omega)|$) (b) Frequency response ($|T(j\omega)|$)

Loop Shaping

Bode's Relations (§ 9.4)

the phase is uniquely given by the shape of the gain curve (minimum phase systems)

$$\arg G(j\omega_0) \approx \frac{\pi}{2} \frac{d \log|G(j\omega)|}{d \log \omega} \quad \text{[rad]} \tag{9.8}$$

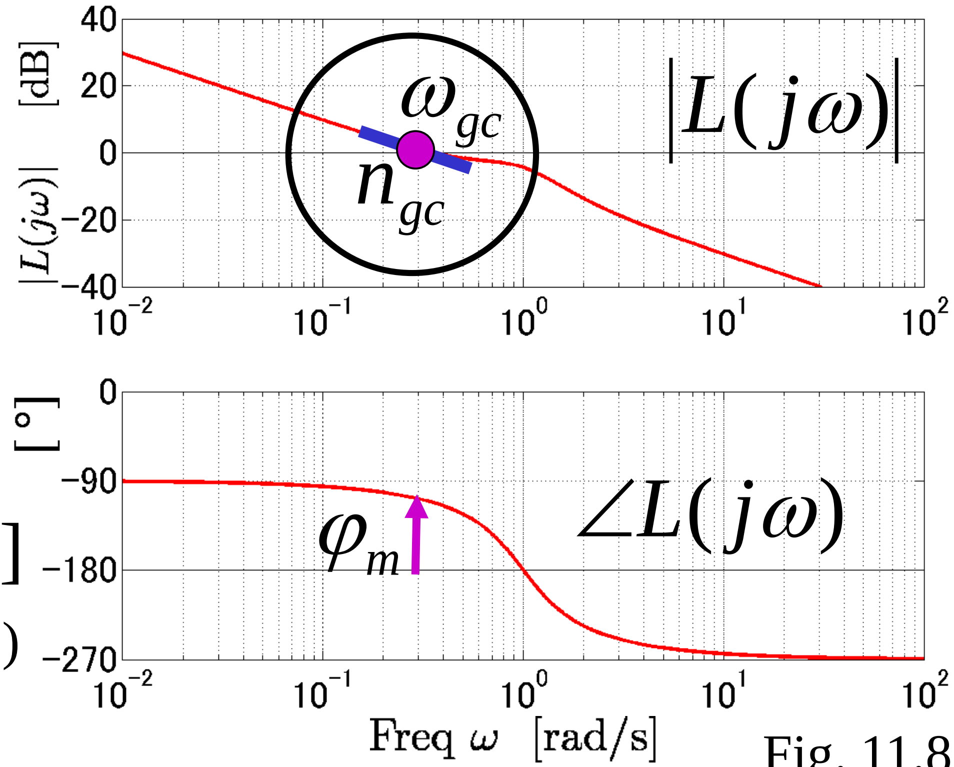


Fig. 11.8

at gain crossover frequency ω_{gc}

$$-\pi + \varphi_m = \frac{\pi}{2} n_{gc}$$

n_{gc} : slope of the gain curve at gain crossover frequency ω_{gc}

Bode's Relations (§ 9.4)

$$-\pi + \varphi_m = \frac{\pi}{2} n_{gc}$$

$$n_{gc} = -2 + \frac{2\varphi_m}{\pi} \qquad (11.11)$$

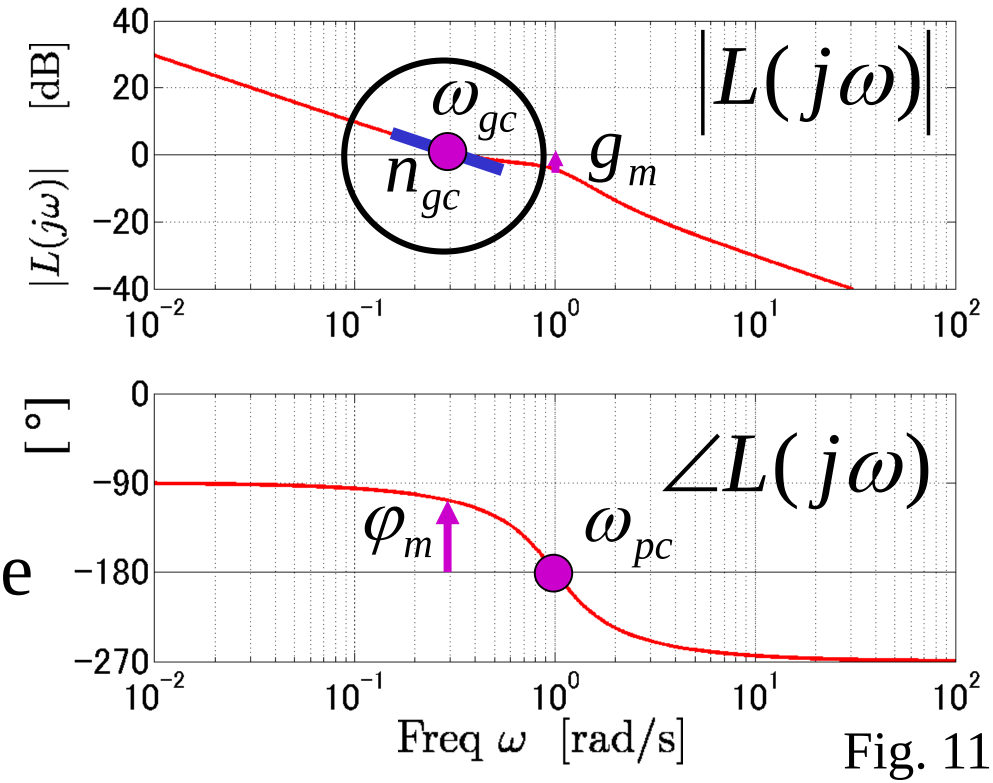


Fig. 11.8

φ_m : phase margin

$(\varphi_m = 30° - 90°)$

n_{gc} : slope of the gain curve at gain crossover ω_{gc}

$$\left(-\frac{5}{3} \le n_{gc} \le -1\right)$$

$$n_{gc} = -1 \rightarrow \varphi_m = \pi/2 \ (90°)$$

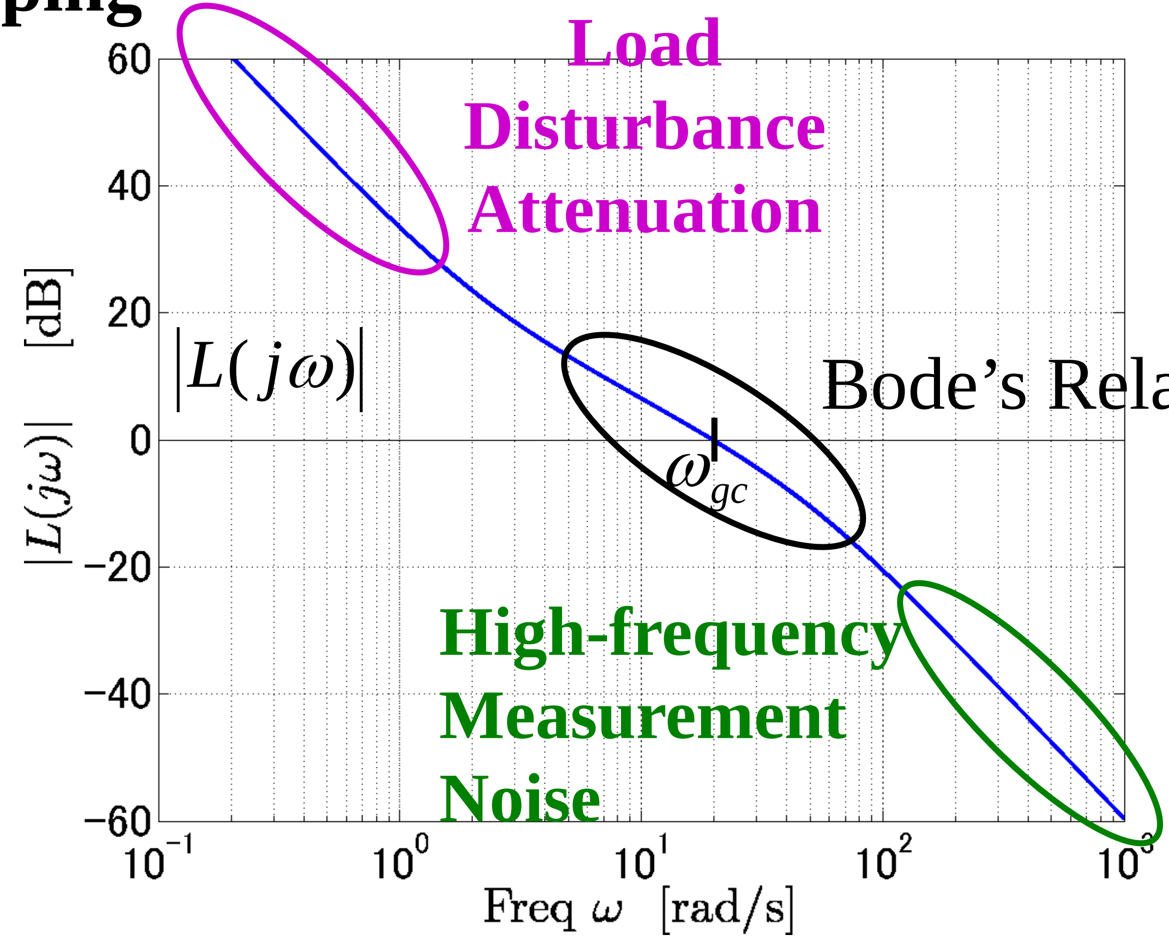
$$n_{gc} = -2 \rightarrow \varphi_m = 0 \ \ (0°)$$

g_m : gain margin

$(g_m = 2 - 5)$

the slope of the gain curve at gain crossover ω_{gc} cannot be too steep

Loop Shaping



Load Disturbance Attenuation

Bode’s Relations

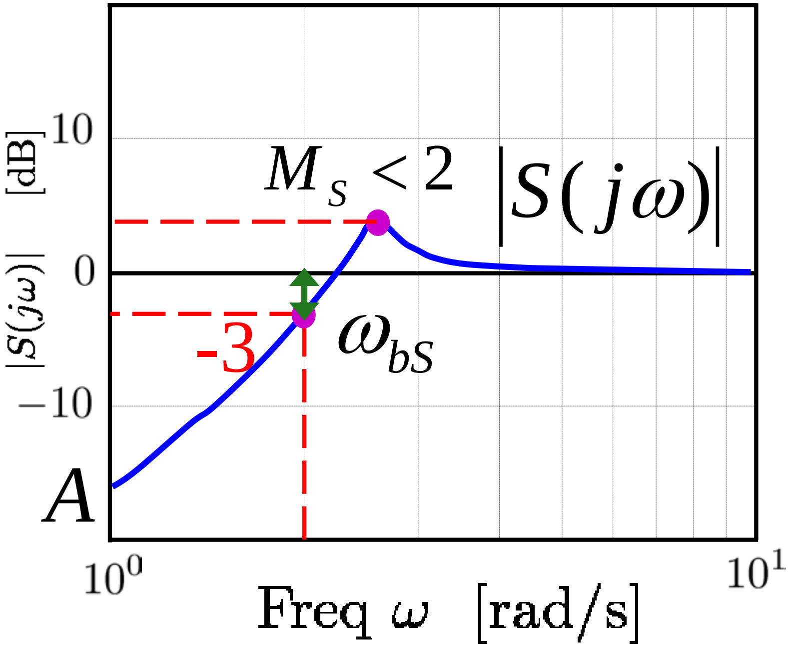
High-frequency Measurement Noise

Fig. 11.8

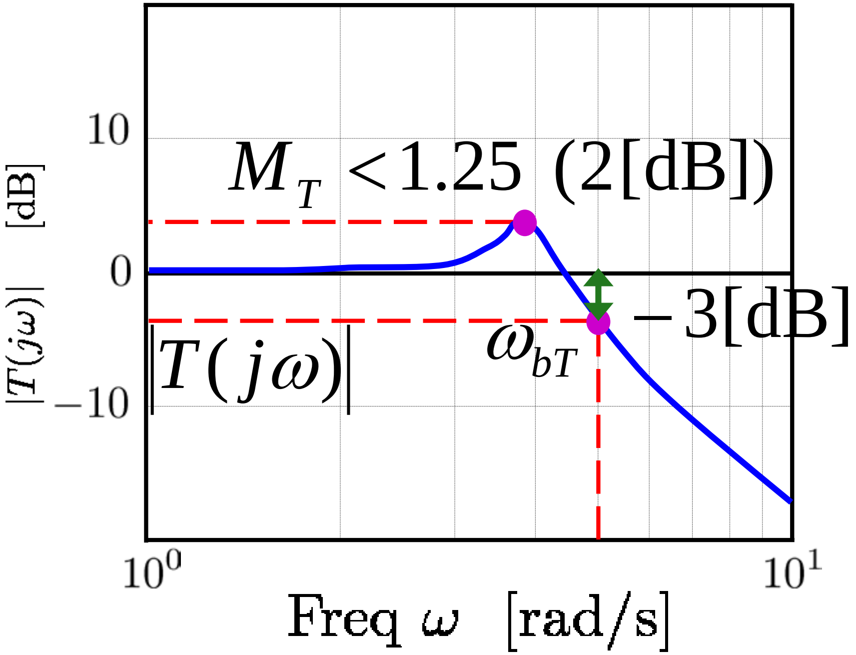
(a) Frequency response ($L(s)$)

- Gain Margin $g_m = 1/|L(i\omega_{pc})|$ $(2-5)$
- Phase Margin $\varphi_m = \pi + \arg L(i\omega_{gc})$ $(30° - 60°)$
- Stability Margin $s_m = 1/M_S$ $(0.5 - 0.8)$

Sensitivity Function $S(j\omega)$



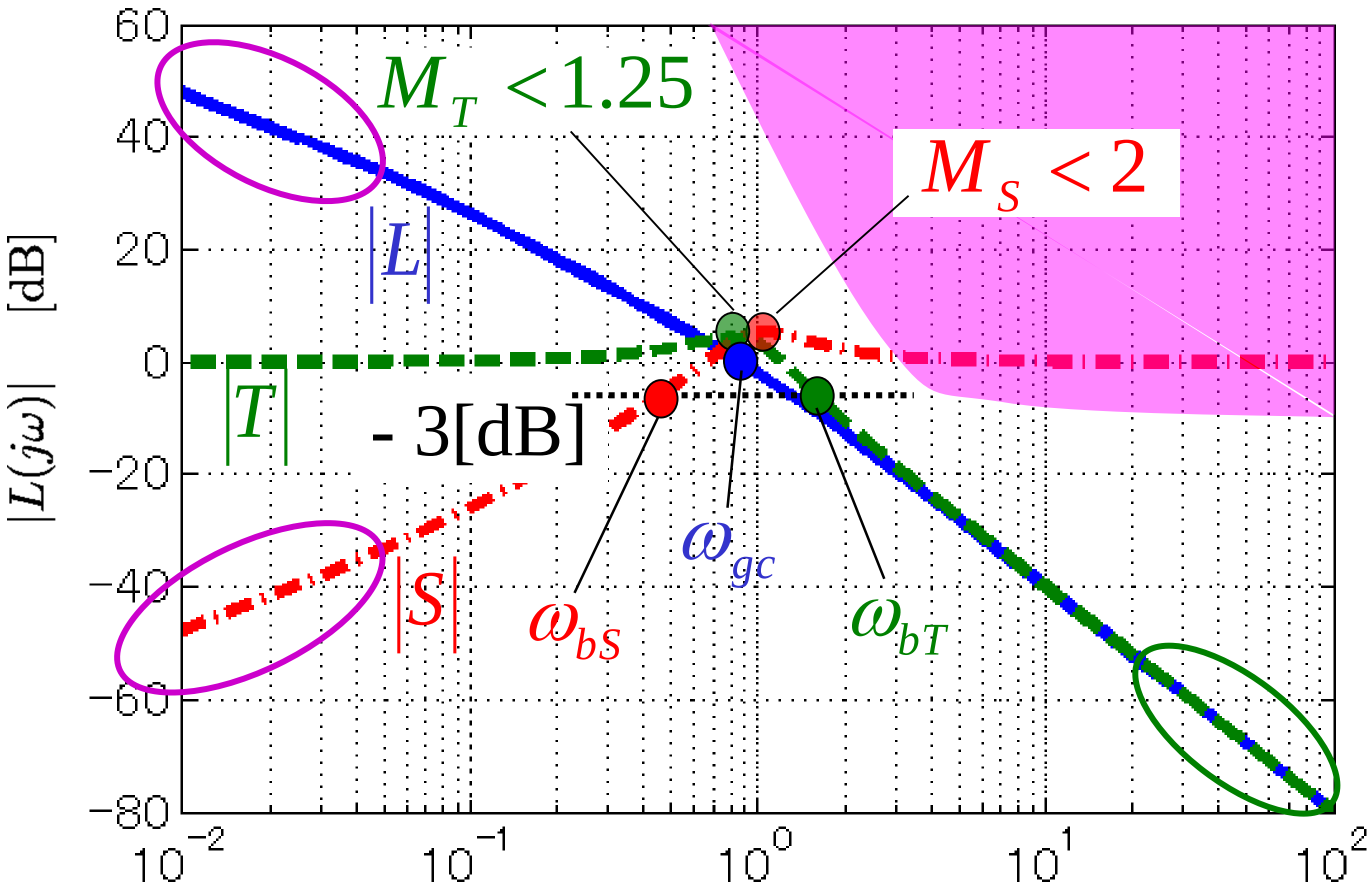
Complementary Sensitivity Function $T(j\omega)$



$$S+T=\frac{1}{1+PC}+\frac{PC}{1+PC}=1$$

- Maximum Peak Magnitude of S $M_S < 2$
- Maximum Peak Magnitude of T $M_T < 1.25$

Loop Shaping



Important Relations

$$g_m \geq \frac{M_S}{M_S - 1} \quad \varphi_m \geq 2\arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S}$$

Exercise

$$g_m \geq 1 + \frac{1}{M_T} \quad \varphi_m \geq 2\arcsin\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T}$$

11.5 Fundamental Limitations

[Ex. 11.9] Balance system (§ 6.3)

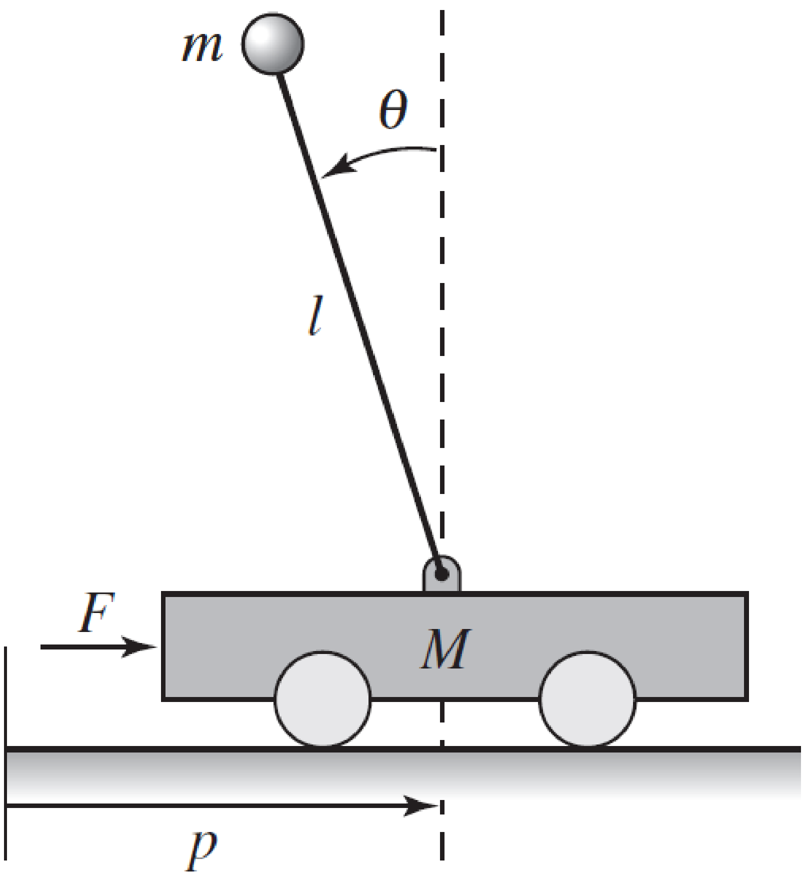


Fig. 6.2 (a) Segway

(b) Cart-pendulum system

Equations of motion

$$\begin{aligned}(M+m)\ddot{p}-ml\cos\theta\ddot{\theta}&=-c\dot{p}-ml\sin\theta\dot{\theta}^2+F\\(J+ml^2)\ddot{\theta}-ml\cos\theta\ddot{p}&=-\gamma\dot{\theta}+mgl\sin\theta\end{aligned}\qquad(6.4)$$

[Ex. 11.9] Balance system (§ 6.3)

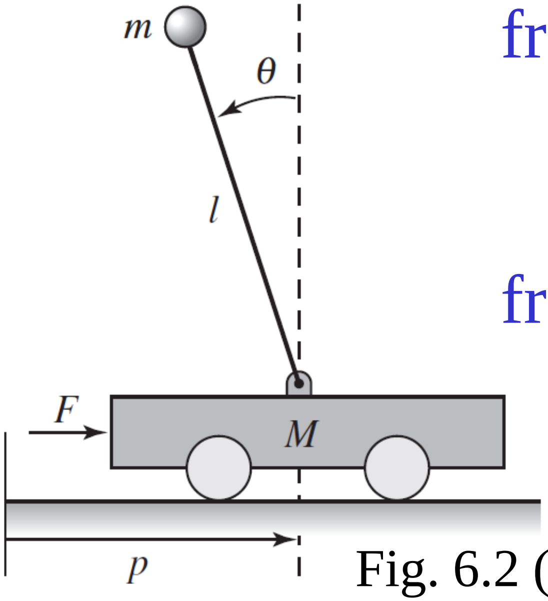


Fig. 6.2 (b)

from F to θ

$$H_{\theta F} = \frac{ml}{-(M_t J_t - m^2 l^2)s^2 + mglM_t}$$

from F to p

$$H_{pF} = \frac{-J_t s^2 + mgl}{s^2(-(M_t J_t - m^2 l^2)s^2 + mglM_t)}$$

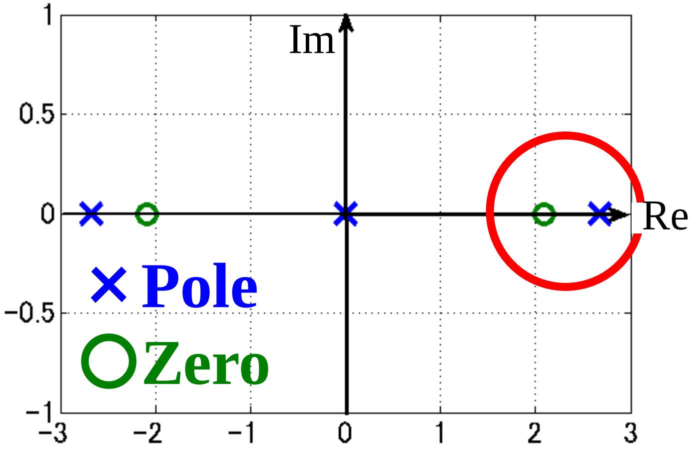
$$* \; M_t = M + m \quad J_t = J + ml^2$$

poles: $\left\{0, 0, \pm\sqrt{mglM_t / \left(M_t J_t - m^2 l^2\right)}\right\}$

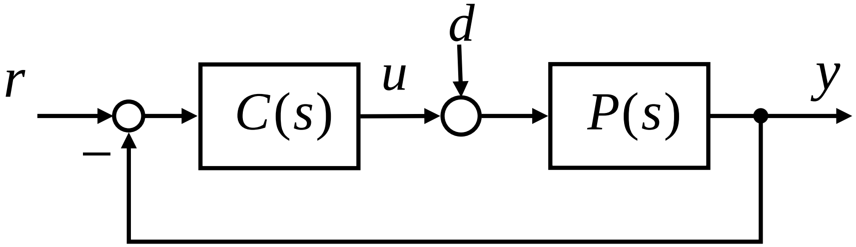
zeros: $\left\{\pm\sqrt{mgl / J_t}\right\}$

H_{pF} : RHP pole $p = 2.68$

RHP zero $z = 2.09$



Effect of RHP Poles



$$P(s)=\frac{y(s)}{u(s)}=\frac{1}{s-1}$$

$$u(s)=C(s)(r(s)-y(s))$$

$$C(s)=\frac{s-1}{s}$$

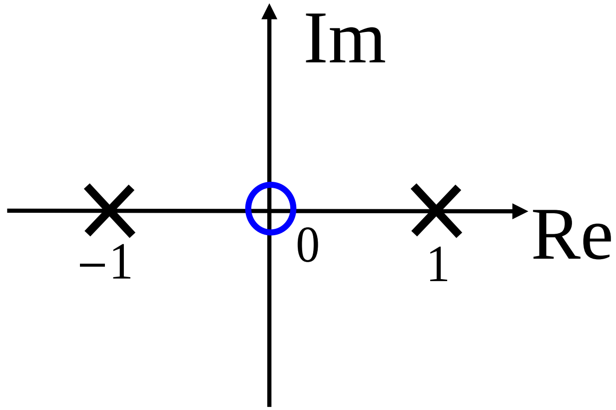
$$y(s)=\frac{1}{s+1}r(s)+\frac{s}{(s+1)\underline{(s-1)}}y_0$$

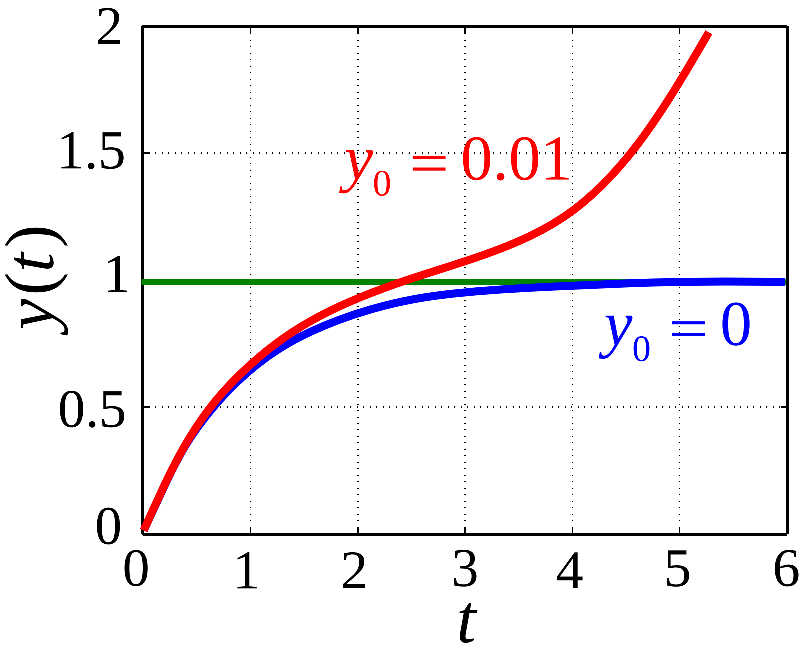
Unstable

y_0 : initial value

Pole（×）: −1, 1

Zero（○）: 0

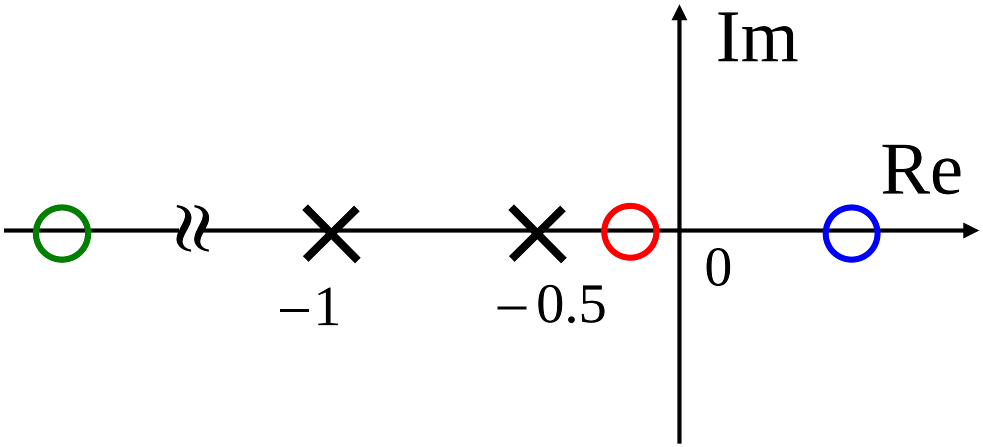




Step response

Effect of RHP Zeros

$$G(s)=\frac{as+1}{(s+1)(2s+1)}$$



Pole (×) : $-1,\ -0.5$

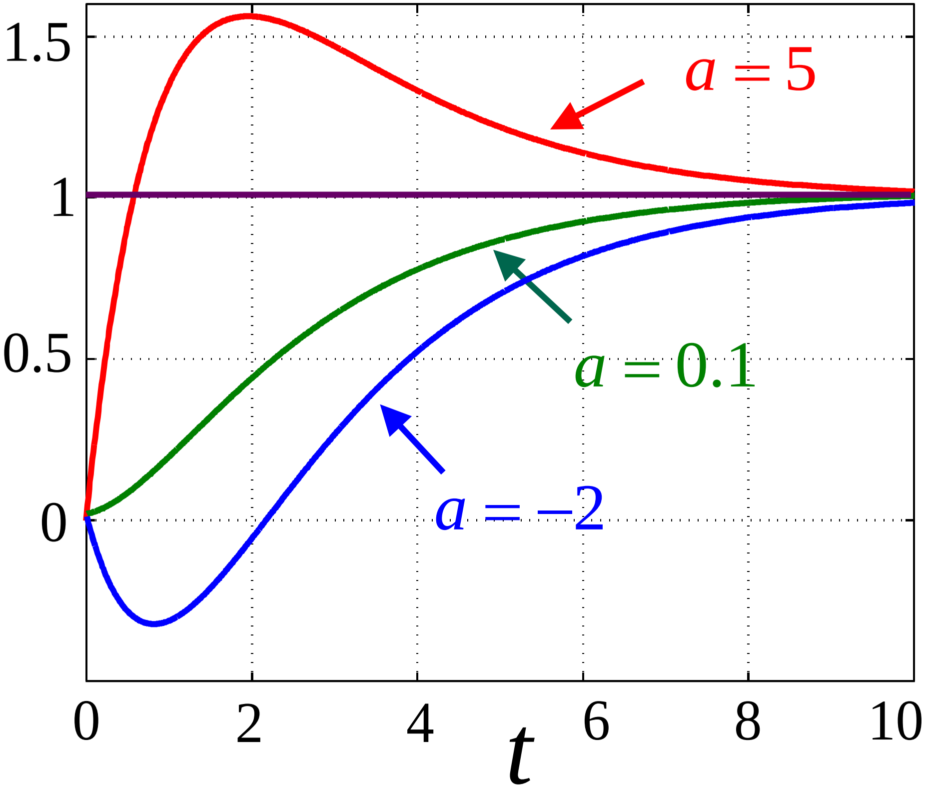
Zero (○) : $-\frac{1}{a}$

a : Small ⇒ No Effect

a : Large ⇒ Overshoot

$a<0$: (Unstable)

⇒ Undershoot



Gain Crossover Frequency Inequality

Factor the process transfer function as

$$P(s) = P_{mp}(s)P_{ap}(s) \qquad (11.13)$$

P_{mp} : minimum phase part

P_{ap} : all-pass system

(nonminimum phase part s.t. $|P_{ap}(j\omega)| = 1$ $\arg P_{ap}$: negative)

RHP poles, zeros and time delay

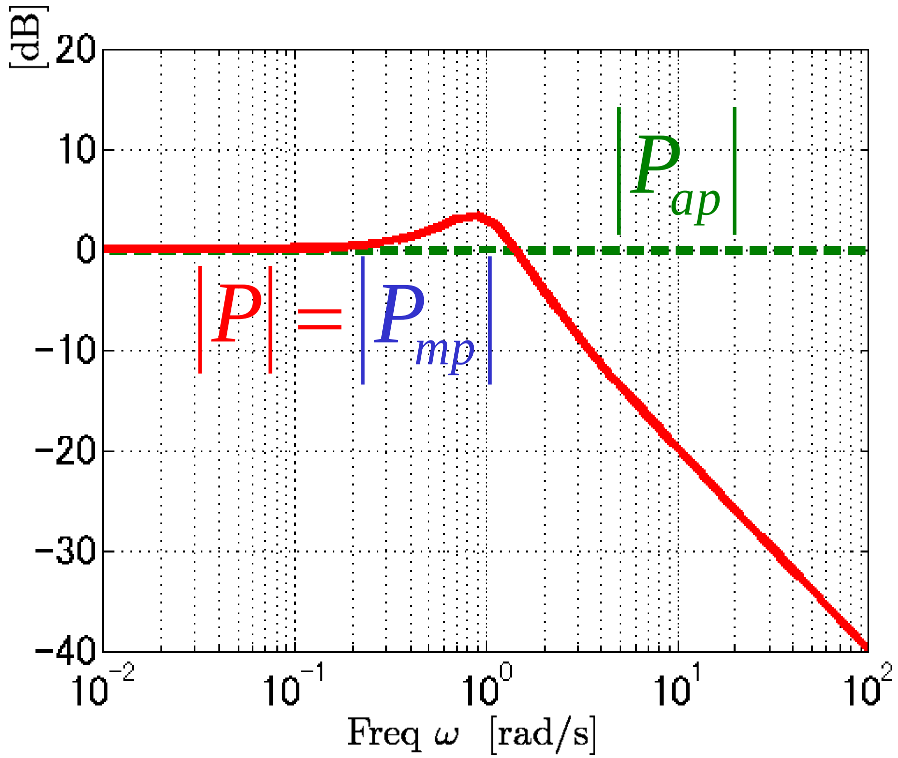
Ex.)

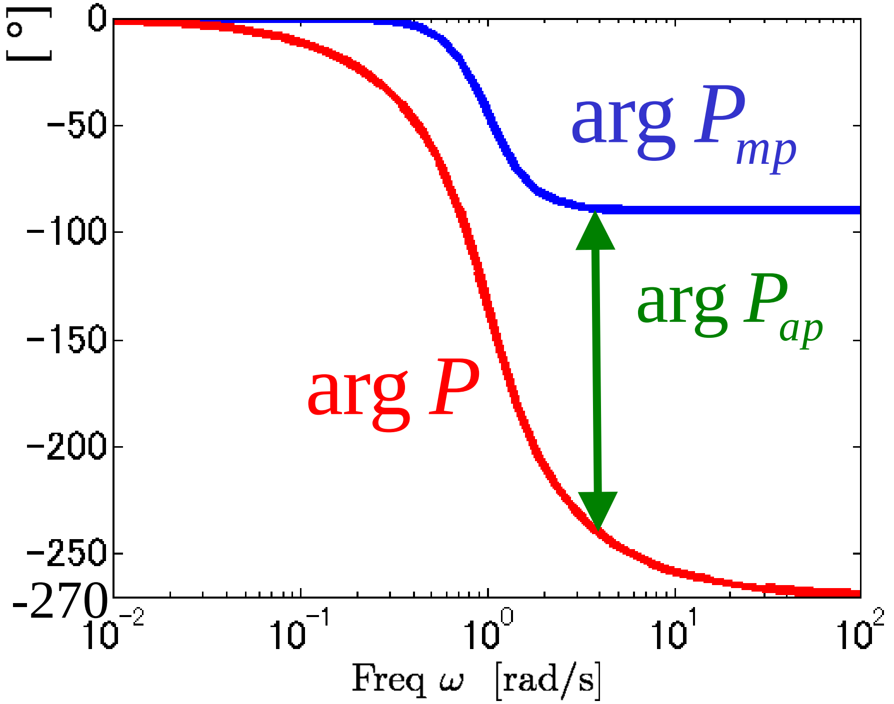
$$P(s) = \frac{1-s}{s^2+s+1} = \underbrace{\frac{1+s}{s^2+s+1}}_{\text{minimum phase part } P_{mp}} \cdot \underbrace{\frac{1-s}{1+s}}_{\text{all-pass system } P_{ap}}$$

$$* \; |P_{ap}(j\omega)| = \left|\frac{1-j\omega}{1+j\omega}\right| = \frac{\sqrt{1^2+(-\omega)^2}}{\sqrt{1^2+\omega^2}} = 1$$

Ex.)

$$P(s) = \underline{\frac{1-s}{s^2+s+1}} = \underbrace{\frac{1+s}{s^2+s+1}}_{\substack{\text{minimum phase part} \\ P_{mp}}} \cdot \underbrace{\frac{1-s}{1+s}}_{\substack{\text{all-pass system} \\ P_{ap}}}$$





Derivation of the Gain Crossover Frequency Inequality

Phase of $L(j\omega_{gc})$

$$* \; L = PC = P_{mp}P_{ap}C$$

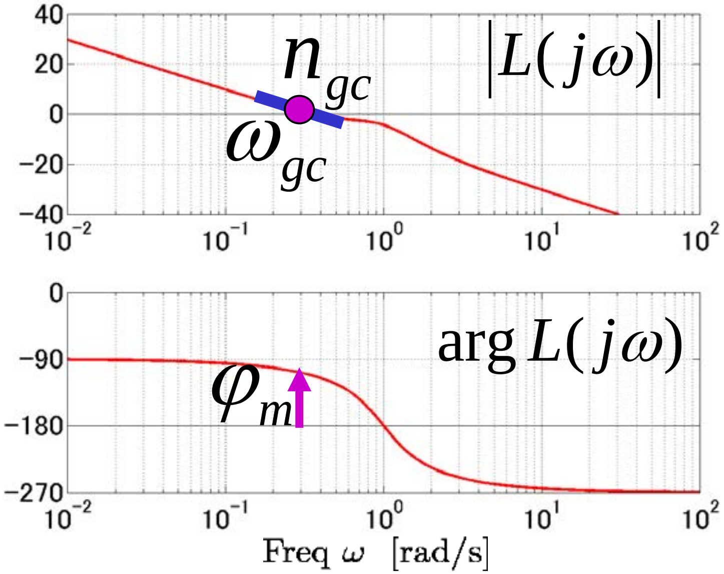
$$\arg L(j\omega_{gc})$$

$$= \arg P_{ap}(j\omega_{gc}) + \arg P_{mp}(j\omega_{gc}) + \arg C(j\omega_{gc})$$

$$\geq -\pi + \varphi_m \tag{11.14}$$

Slope of $\left|L(j\omega_{gc})\right|$ at ω_{gc}

$$n_{gc} = \left.\frac{d\log\left|L(j\omega)\right|}{d\log\omega}\right|_{\omega=\omega_{gc}}$$



$$= \left.\frac{d\log\left|P_{mp}(j\omega)C(j\omega)\right|}{d\log\omega}\right|_{\omega=\omega_{gc}}$$

$$* \quad \left|P_{ap}(j\omega)\right| = 1$$

Derivation of the Gain Crossover Frequency Inequality

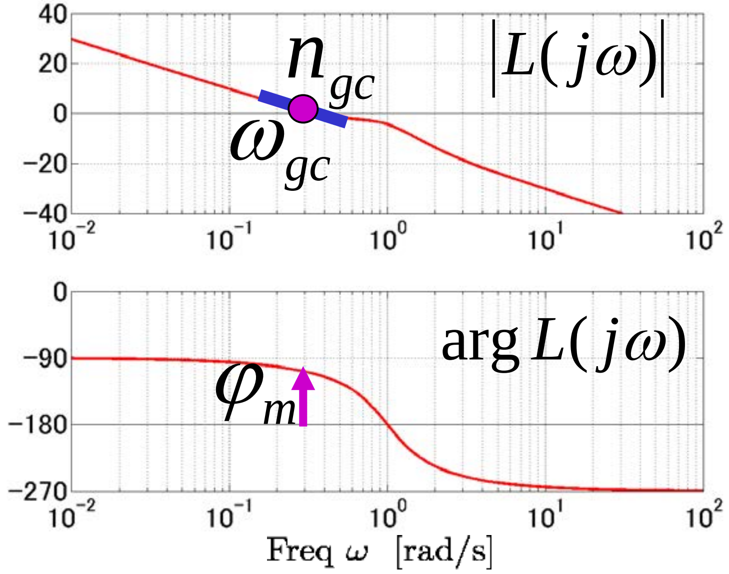
Bode's Relations $$\arg G(j\omega_0) \approx \frac{\pi}{2}\frac{d\log\left|G(j\omega)\right|}{d\log\omega} \quad (9.8)$$

holds for minimum phase systems

$$\arg\left(P_{mp}(j\omega_{gc})C(j\omega_{gc})\right)$$

$$\approx \frac{\pi}{2}\frac{d\log\left|P_{mp}(j\omega_{gc})C(j\omega_{gc})\right|}{d\log\omega_{gc}}$$

$$= n_{gc}\frac{\pi}{2}$$



Combining it with (11.14)

$$\arg P_{ap}(j\omega_{gc}) + \underbrace{\arg P_{mp}(j\omega_{gc}) + \arg C(j\omega_{gc})}_{\arg\left(P_{mp}(j\omega_{gc})C(j\omega_{gc})\right) \approx n_{gc}\frac{\pi}{2}} \geq -\pi + \varphi_m \quad (11.14)$$

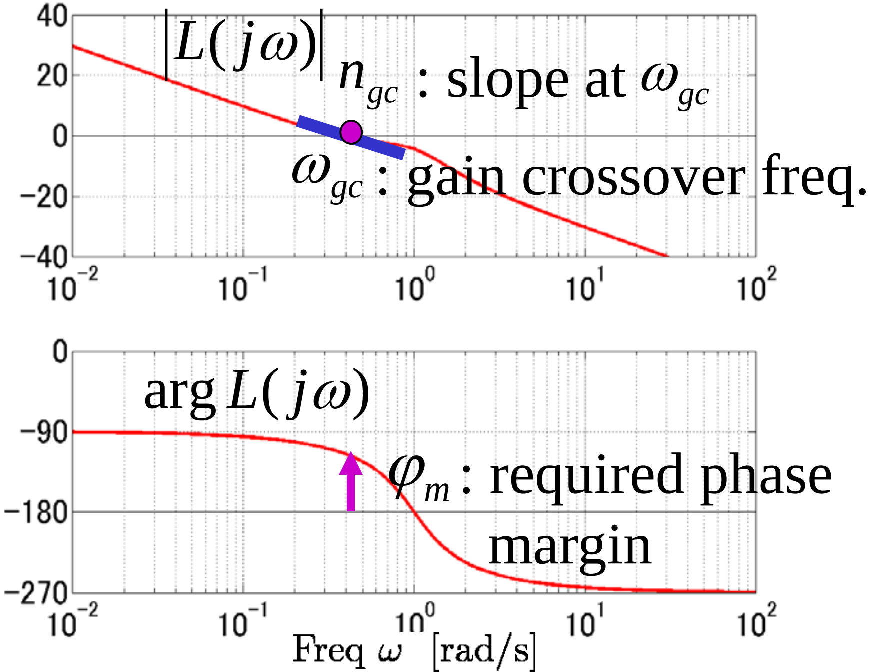
$$-\arg P_{ap}(j\omega_{gc}) \leq \pi - \varphi_m + n_{gc}\frac{\pi}{2} =: \varphi_l \quad (11.15)$$

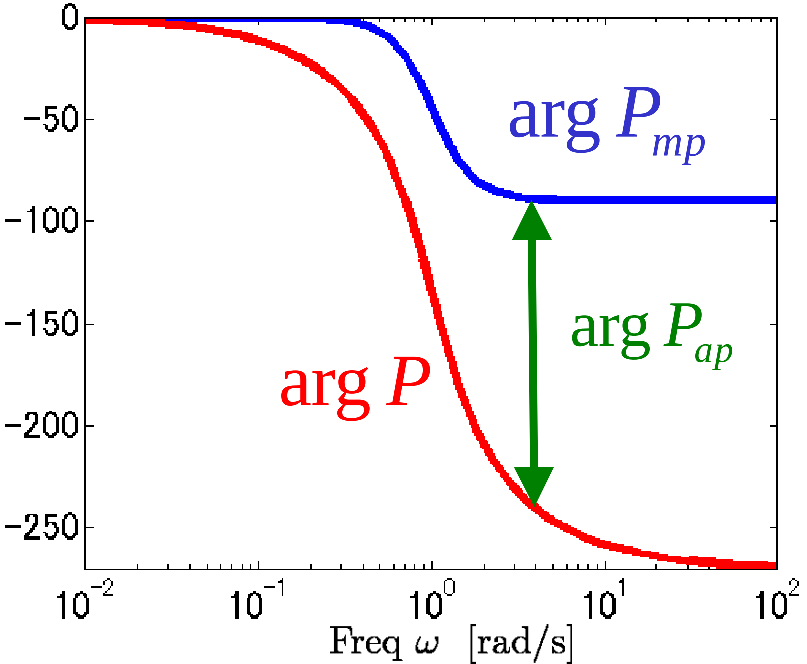
Gain Crossover Frequency Inequality

Gain Crossover Frequency Inequality

$$-\arg P_{ap}(j\omega_{gc}) \le \pi - \varphi_m + n_{gc}\frac{\pi}{2} =: \varphi_l \quad (11.15)$$

- The phase lag of the nonminimum phase component must not be too large at the crossover frequency.
- Nonminimum phase components imposes severe restrictions on possible crossover frequencies.





Gain Crossover Frequency Inequality

Gain Crossover Frequency Inequality

$$-\arg P_{ap}(j\omega_{gc}) \le \pi - \varphi_m + n_{gc}\frac{\pi}{2} =: \varphi_l \qquad (11.15)$$

allowable phase lag of P_{ap} at ω_{gc}: φ_l $\quad \varphi_m = 30^\circ - 60^\circ$

- for high robustness

  required phase margin : $\varphi_m = 60^\circ$

  slope: $n_{gc} = -1$

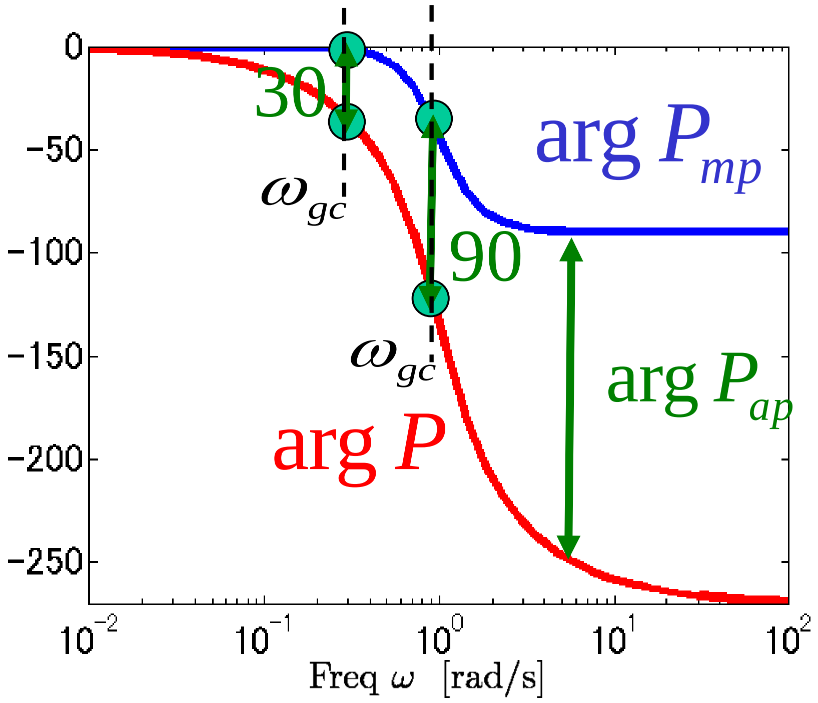
  $\longrightarrow \quad \varphi_l = 30^\circ$

- for lower robustness

  required phase margin : $\varphi_m = 45^\circ$

  slope : $n_{gc} = -1/2$

  $\longrightarrow \quad \varphi_l = 90^\circ$



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