Robust and Optimal Control, Spring 2015

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G. Sum of Squares (SOS)

G.1 SOS Program: SOS/PSD and SDP

G.2 Duality, valid inequalities and Cone

G.3 Feasibility/Optimization and Ideal

G.4 Exploiting Structure

Reference:

Semidefinite Optimization and Convex Algebraic Geometry,
Overview

Feasibility/Optimization Problems

Polynomial Descriptions

1. Non-negativity
   - P-satz Relaxations
   - Infeasibility Certificates

2. PSD cones (SOS Decomposition)
   - SOS/SDP Approach
   - Exploit Structure

3. Convex Optimization
   - (Quadratic Forms)

Semidefinite Programs (SDP)

Solvable in polynomial time
Motivating Examples
Discrete Problems: LQR with Binary Inputs

**System**  Linear discrete-time system

\[ x(t + 1) = Ax(t) + Bu(t) \quad \forall t = 0, 1, \cdots, N, \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \]

\[ y(t) = x(t) \]

\[ u_i(t) \in \{-1, 1\} \; \text{(binary inputs)} \quad \forall i = 1, \cdots, m \quad \forall t = 0, \cdots, N - 1 \]

**Objective**

Given \( x(0) = x_0 \) and the evolution of reference signals \( r(t) \), find an optimal controller \( u \) minimizing the quadratic tracking error

\[
\min_u \sum_{t=0}^{N-1} (x(t) - r(t))^T Q (x(t) - r(t))
\]
Motivating Examples

Nonlinear Problems: Lyapunov Stability

[Ex.] Moore-Greitzer model of a jet engine with controller:

\[
\begin{align*}
\dot{x} &= -y + 1.5x^2 - 0.5x^3 \\
\dot{y} &= 3x - y =: u
\end{align*}
\]

Find a Lyapunov function \( V(x, y) \).

Candidate  4th order polynomial function

\[
V(x, y) = \sum_{0 \leq j + k \leq 4} c_{jk} x^j y^k
\]

Graph Problems: MAX CUT partitioning

Partition the nodes of a graph in two disjoint sets, maximizing the number of edges between sets

How to compute bounds, or exact solutions, for this kind of problems?
Polynomial Programming: Optimization Problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \quad \text{(polynomials)} \\
\text{subject to} & \quad f_i(x) \leq 0 \quad \text{(polynomials)} \quad i = 1, \cdots, m \\
& \quad h_i(x) = 0 \quad \text{(polynomials)} \quad i = 1, \cdots, p
\end{align*}
\]

[Ex.] Given \( f \in \mathbb{R}[x_1, \cdots, x_n] \),

Primal decision problem

(\( P \)) \( \exists x \in \mathbb{R}^n \text{ s.t. } f(x) < 0 \quad ? \\
\downarrow \text{ No}

(\( \not= P \)) \( f(x) \geq 0 \quad \forall x \in \mathbb{R}^n \)

\( f \) : globally non-negative

Polynomial non-negativity

(i.e., \( f \) is positive definite: PSD)

This problem is \textbf{NP-hard}

But, decidable.

Certificates:

\textbf{Yes} \quad Exhibit \( x \) \text{ s.t. } f(x) < 0 \\
\textbf{No} \quad Need a certificate/witness

i.e., a proof that there is no feasible point

(Infeasibility certificate)
Sum of Squares (SOS) Decomposition

For \( x \in \mathbb{R}^n \), a multivariate polynomial \( f(x) \) is a *sum of squares* if there exist some polynomials \( g_1, \cdots, g_s \in \mathbb{R}[x_1, \cdots, x_n] \) such that \( f(x) = \sum_{i=1}^{s} g_i^2(x) \). Then, \( f(x) \) is nonnegative.

We can write any polynomial as a quadratic function of monomials

[Ex.] \[
f(x, y) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4
\]

\[
= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7 + 2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} = z^T Q(\lambda) z \quad \forall \lambda \in \mathbb{R}
\]

If for some \( \lambda \), we have \( Q(\lambda) \succeq 0 \), then we can factorize \( Q(\lambda) \)

\[
Q(6) = \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \quad \Rightarrow \quad f = \left\| \begin{array}{c} 2xy + y^2 \\ 2x^2 + xy - 3y^2 \end{array} \right\|^2 = \left\| \begin{array}{c} g_1(x, y) \\ g_2(x, y) \end{array} \right\|^2
\]

\[
f(x, y) = z^T Q(6) z \quad \Leftrightarrow \quad (\neq P) \quad f(x) \geq 0 \quad \forall x \in \mathbb{R}^n
\]

\( f \) : Polynomial non-negativity
Suppose \( f \in \mathbb{R}[x_1, \cdots, x_n] \), of degree \( 2d \)
Let \( z \) be a vector of all monomials of degree less than or equal to \( d \)

\[
f(x) \text{ is SOS iff } \exists Q \text{ such that } Q \succeq 0 \text{ and } f = z^T Q z \quad \text{[SDP]}
\]

The number of components of \( z \) is \( n + d C_d = \frac{(n + d)!}{n! d!} \)

Comparing terms gives affine constraints on the elements of \( Q \)

If \( Q \) is a feasible point of the SDP, then to construct the SOS representation

Factorize \( Q = VV^T \), and write \( V = [v_1 \cdots v_r] \), so that

\[
f = z^T V V^T z = \|V^T z\|^2 = \sum_{i=1}^{r} (v_i^T z)^2
\]

One can factorize using e.g., Cholesky or eigenvalue decomposition

The number of squares \( r \) equals the rank of \( Q \)
Convexity

The set of PSD and SOS polynomials are a *convex cones*, i.e., \( f, g \) are PSD \( \Rightarrow \lambda f + \mu g \) is PSD \( \forall \lambda, \mu \geq 0 \)

Let \( P_{n,d} \) be the set of SPD polynomials of degree \( \leq d \)
Let \( \Sigma_{n,d} \) be the set of SOS polynomials of degree \( \leq d \)
Both \( P_{n,d} \) and \( \Sigma_{n,d} \) are convex cones in \( \mathbb{R}^n \) where \( N = n + dC_d \)
We know \( \Sigma_{n,d} \subset P_{n,d} \), and testing if \( f \in P_{n,d} \) is NP-hard
But testing if \( f \in \Sigma_{n,d} \) is an SDP (but a large one)

\[
\begin{align*}
\text{PSD} = \text{SOS} & \iff \\
(i) & \quad d = 2 : \text{quadratic polynomials} \\
(ii) & \quad n = 1 : \text{univariate polynomials} \\
(iii) & \quad d = 4, n = 2 : \text{quartic polynomials in two variables}
\end{align*}
\]

In general, \( f \) is PSD does not imply \( f \) is SOS

David Hilbert

Every PSD polynomial is a SOS of *rational functions*
Why does this work?

Three independent facts, theoretical and experimental:

1. The existence of efficient algorithms for SDP
2. The size of the SDPs grows much slower than the Bezout number $\mu$
   -- A bound on the number of (complex) critical points
   -- A reasonable estimate of complexity
   -- $\mu = (2d - 1)^n$ (for dense polynomials)
   -- Almost all (exact) algebraic techniques scale as $\mu$
3. The lower bound $f^{SOS}$ very often coincides with $f^*$
   -- Why? What does often mean?

SOS provides short proofs, even though they are not guaranteed to exist.
Help on SOS

U. Topcu, A. Packard, P. Seiler, G. Balas,

MATLAB Toolbox: SOS TOOLS

(with MATLAB symbolic toolbox, SeDuMi)
http://www.cds.caltech.edu/sostools/
Diagram Depicting Relations

1) Initialize a SOS Program, declare the SOS program variable
2) Define SOS program constraints
3) Set objective function (for optimization problem)
4) Call solver
5) Get solutions
SOS Problem: Example of SOSTOOLS

Problem sosdemo2.m

Find a polynomial

\[ V(x) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 \]

( \( a_i \) : the unknown decision variables)

Constraints

\[ V(x) - (x_1^2 + x_2^2 + x_3^2) \geq 0 \]
\[ -(x_3^2 + 1) \left( \frac{\partial V}{\partial x_1} \dot{x}_1 - \frac{\partial V}{\partial x_2} \dot{x}_2 - \frac{\partial V}{\partial x_3} \dot{x}_3 \right) \geq 0 \]

Solution

\[ V(x) = 7.1525 x_1^2 + 5.7870 x_2^2 + 2.1434 x_3^2 \]
Problem: \[ \min_{x, y} F(x, y) \]
with \[ F(x, y) := 4x^2 - \frac{21}{10} x^4 + \frac{1}{3} x^6 + xy - 4y^2 + 4y^4 \]

Not convex. Many local minima. NP-hard. How to find good lower bounds?

Find the largest \( \gamma \) s.t.

\[ F(x, y) - \gamma \text{ is SOS} \]

If exact, can recover optimal solution

Surprisingly effective

Solving, the maximum \( \gamma \) is -1.0316. Exact bound.

SOS program: Coefficient Space

Problem: Let \( f_{\alpha \beta}(x) = x^4 + (\alpha + 3\beta)x^3 + 2\beta x^2 - \alpha x + 1 \)
What is the set of values of \((\alpha, \beta) \in \mathbb{R}^2\) for which \( f_{\alpha \beta} \) is PSD? SOS?

SOS decomposition

\[
f_{\alpha \beta}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}
\]

\[= q_{33} x^4 + 2q_{23} x^3 + (q_{22} + 2q_{13}) x^2 + 2q_{12} x + q_{11} \]

Feasible set (satisfying PSD)

\[
\begin{cases}
(\alpha, \beta) & \exists \lambda \text{ s.t. } \\
\begin{bmatrix}
1 & - (1/2)\alpha & \beta - \lambda \\
-(1/2)\alpha & 2\lambda & (1/2)(\alpha + 3\beta) \\
\beta - \lambda & (1/2)(\alpha + 3\beta) & 1
\end{bmatrix} \succeq 0
\end{cases}
\]

\( f_{\alpha \beta} \): univariate polynomials \( \rightarrow \) PSD = SOS

Convex and semi-algebraic
SOS program: Lyapunov Stability Analysis

To prove asymptotic stability of $\dot{x} = f(x)$ at $x = 0$

- $V(0) = 0, \dot{V}(0) = 0,$
- $V(x) > 0, \dot{V}(x) = \nabla V(x) \cdot f(x) < 0 \ \forall x \neq 0$

A. Lyapunov

[Ex.] $\dot{x} = Ax \quad \rightarrow \quad V(x) = x^T P x \quad \text{s.t.} \quad P > 0, \quad A^T P + PA < 0$

Check nonnegativity
(affine of quadratic forms: LMI)

Check SOS conditions
(affine of polynomials)

[Ex.] Moore-Greitzer model of a jet engine with controller

Find a $V(x, y)$ s.t. $V(x, y)$ is SOS and $-\dot{V}(x, y)$ is SOS

Both conditions are affine in the coefficients, so can use SOS/SDP

Resulting Lyapunov function

$$V = 4.5819x^2 - 1.5786xy + 1.7834y^2$$
$$-0.12739x^3 + 2.5189x^2y - 0.34069xy^2$$
$$+0.61188y^3 + 0.47537x^4 - 0.052424x^3y$$
$$+0.44289x^2y^2 + 1.8868 \times 10^{-6}xy^3 + 0.090723y^4$$
SOS program: Lyapunov Stability Analysis

[Ex.] Autonomous System

\[
\begin{align*}
\dot{x} &= -x + (1 + x)y \\
\dot{y} &= -(1 + x)x
\end{align*}
\]

For the system, we can find a Lyapunov function with quartic polynomial

\[
V(x, y) = 6x^2 - 2xy + 8y^2 - 2y^3 + 3x^4 + 6x^2y^2 + 3y^4
\]

\[
V(x, y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix}
6 & -1 & 0 & 0 & 0 \\
-1 & 8 & 0 & 0 & -1 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 \\
0 & -1 & 0 & 0 & 3
\end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}
\]

is SOS

\[
-\dot{V}(x, y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}^T \begin{bmatrix}
10 & 1 & -1 & 1 \\
1 & 2 & 1 & -2 \\
-1 & 1 & 12 & 0 \\
1 & -2 & 0 & 6
\end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}
\]

is SOS

The matrices are positive definite, so this proves asymptotic stability.
Nonlinear Control Synthesis

Lyapunov stability criterion (asymptotic stability)

For \( \dot{x} = f(x) \), a Lyapunov function \( V(x) \) satisfies

\[
V(0) = 0 \ , \ V(x) > 0, \ x \neq 0 \quad \left( \frac{\partial V}{\partial x} \right)^T f(x) < 0, \ \forall x \neq 0
\]

\[\uparrow\]

“Dual”

Anders Rantzer

a “dual” Lyapunov function: \( \nabla \cdot (\rho f) > 0 \)

The synthesis problem is convex in \( (\rho, u\rho) : \nabla \cdot [\rho(f + gu)] > 0 \)

Parametrizing \( (\rho, u\rho) \), can apply SOS methods

[Ex.] \[
\begin{align*}
\dot{x} &= y - x^3 + x^2 \\
\dot{y} &= u
\end{align*}
\]

A stabilizing controller is:

\[
u(x, y) = -1.22x - 0.57y - 0.129y^3
\]

Summary: About SOS/SDP

Semi-definite matrices are SOS quadratic forms
SOS polynomials are embedded into PSD cone

\[ f(x) \text{ is SOS iff } \exists Q \text{ such that } Q \succeq 0 \text{ and } f = z^T Q z \] [SDP]

The resulting SDP problem is polynomially sized (in \( n \), for fixed \( d \))

By properly choosing the monomials, we can exploit structure
(sparisity, symmetries, ideal structure, graph structure, etc.)

Important Feature: The problem is still a SDP
if the coefficients of \( F \) are variable, and the dependence is affine

Can optimize over SOS polynomials in affinely described families

[Ex.] If \( p(x) = p_0(x) + \alpha p_1(x) + \beta p_2(x) \),
we can “easily” find values of \( \alpha, \beta \) for which \( p(x) \) is SOS

This fact (Exploiting this structure) will be crucial in applications
Dual Problem: Motivating Example

Primal Problem:  \( \min x_1 x_2 \)
subject to  \( x_1 \geq 0 \quad x_2 \geq 0 \)
\( x_1^2 + x_2^2 \leq 1 \)

Dual Problem:  \( \max g(\lambda) \)
subject to  \( \lambda_1 \geq 0 \quad \lambda_2 \geq 0 \)
\( \lambda_3 \geq 1/2 \)

Lagrange dual function:
\[
g(\lambda) = \inf_{x} x_1 x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3 (x_1^2 + x_2^2 - 1)
\]
\[
= \begin{cases} 
-\lambda_3 - \frac{1}{2} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \begin{bmatrix} 2\lambda_3 & 1 \\ 1 & 2\lambda_3 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} & \text{if } \lambda_3 > 1/2 \\
-\infty & \text{otherwise, except boundary condition}
\end{cases}
\]

Non-convex

Duality Gap

Optimal  \( g(\lambda^*) = -1/2 \)
\( \text{at } \lambda^* = (0, 0, 1/2) \)
The Dual is not intrinsic

The dual problem, and its corresponding optimal value, are not properties of the primal feasible set and objective function alone.

(Instead, they depend on the particular equations and inequalities used)

To construct equivalent primal optimization problems with different duals:

(1) Replace the objective $f_0(x)$ by $h(f_0(x))$ where $h$ is increasing

(2) Introduce new variables and associated constraints, e.g.,

\[
\begin{align*}
\text{minimize} &\quad (x_1 - x_2)^2 + (x_2 - x_3)^2 \\
\text{minimize} &\quad (x_1 - x_2)^2 + (x_4 - x_3)^2 \\
\text{subject to} &\quad x_2 = x_4
\end{align*}
\]

(3) Add redundant constraints
Recall: Motivating Example

Primal Problem': \[ \text{min } x_1 x_2 \]
subject to \[ x_1 \geq 0, \quad x_2 \geq 0 \]
\[ x_1^2 + x_2^2 \leq 1 \]
\[ x_1 x_2 \geq 0 \]
(adding the redundant constraint)

\[
\begin{align*}
\text{Dual Problem: } & \max g(\lambda) \\
\text{subject to } & \lambda_1 \geq 0, \quad \lambda_2 \geq 0 \\
& 2\lambda_3 \geq 1 - \lambda_4 \\
& \lambda_4 \geq 0
\end{align*}
\]

This problem may be written as an SDP using Schur complement

Optimal \[ g(\lambda^*) = 0 \]
at \[ \lambda^* = (0, 0, 0, 1) \]

\[
g(\lambda) = \inf_x x_1 x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3 (x_1^2 + x_2^2 - 1) - \lambda_4 x_1 x_2
\]

\[
= \begin{cases} 
-\lambda_3 - \frac{1}{2} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \begin{bmatrix} 2\lambda_3 & 1 - \lambda_4 \\ 1 - \lambda_4 & 2\lambda_3 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} & \text{if } 2\lambda_3 > 1 - \lambda_4 \\
-\infty & \text{otherwise, except boundary condition}
\end{cases}
\]

Adding redundant constraints makes the dual bound \textbf{tighter}
This always happens! Such constraints are called \textbf{valid inequalities}
Algebraic Geometry

There is a correspondence between the geometric object (the feasible subset of $\mathbb{R}^n$) and the algebraic object (the cone of valid inequalities)

This is a dual relationship

The dual problem is constructed from the cone

For equality constraints, there is another algebraic object; the ideal generated by the equality constraints

For optimization, we need to look both at the geometric objects (for the primal) and the algebraic objects (for the dual problem)
An Algebraic Approach to Duality

Feasibility Problem:
\[ \exists x \in \mathbb{R}^n \text{ s.t. } f_i(x) \geq 0 \text{ (polynomials)} \quad i = 1, \ldots, m \]

Primal Problem”:
Given \( t \in \mathbb{R} \), \( \exists x \in \mathbb{R}^2 \) s.t.
\[ x_1 x_2 \leq t \quad x_1 \geq 0 \quad x_2 \geq 0 \quad x_1^2 + x_2^2 \leq 1 \]

\[ S \neq \emptyset \text{ where } \]
\[ S = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0 \, \forall i = 1, \ldots, m \} \]
with
\[ f_1(x) = t - x_1 x_2 \]
\[ f_2(x) = 1 - x_1^2 - x_2^2 \]
\[ f_3(x) = x_1 \]
\[ f_4(x) = x_2 \]

Optimal of dual problem \( g(\lambda^*) = -1/2 \) at \( \lambda^* = (0, 0, 1/2) \)
An Algebraic Approach to Duality

Every polynomial in \( \text{cone}\{f_1, \cdots , f_m\} \) is nonnegative on the feasible set

\[
\text{If there is a polynomial } q \in \text{cone}\{f_1, \cdots , f_m\} \text{ which satisfies } \quad q(x) \leq -\epsilon < 0 \quad \forall x \in \mathbb{R}^n 
\]

then the primal problem is infeasible.

[Ex.] (Cont.) Let \( q(x) = f_1(x) + \frac{1}{2} f_2(x) \).

Then clearly \( q \in \text{cone}\{f_1, f_2, f_3, f_4\} \) and

\[
q(x) = t - x_1 x_2 + \frac{1}{2} (1 - x_1^2 - x_2^2) = t + \frac{1}{2} - \frac{1}{2} (x_1 + x_2)^2 \leq t + \frac{1}{2}
\]

So for any \( t < -\frac{1}{2} \), the primal problem is infeasible.

This corresponds to Lagrange multipliers \((1, \frac{1}{2})\) for the theorem of alternatives.

Alternatively, this is a proof by contradiction

\[
\begin{align*}
\text{If there exists } x \text{ such that } f_i(x) \geq 0, i = 1, \cdots , 4 \text{ then we must also have } q(x) \geq 0, \text{ since } q \in \text{cone}\{f_1, f_2, f_3, f_4\} \\
\text{But we proved that } q \text{ is negative if } t < -\frac{1}{2}
\end{align*}
\]
An Algebraic Approach to Duality

[Ex.] (Cont.) Let $q(x) = f_1(x) + f_3(x)f_4(x) = t$. giving the stronger result that for any $t < 0$ the inequalities are infeasible. Again, this corresponds to Lagrange multipliers $(1, 1)$.

In both of these examples, we found $q$ in the cone which was globally negative. We can view $q$ as the Lagrange function evaluated at a particular value of $\lambda$.

The Lagrange multiplier procedure is searching over a particular subset of functions in the cone; those which are generated by linear combinations of the original constraints.

By searching over more functions in the cone we can do better.

**Normalization**

$q(x) = f_1(x) + \frac{1}{2}f_2(x) = t + \frac{1}{2} - \frac{1}{2}(x_1 + x_2)^2$

We can also show that $-1 \in \text{cone}\{f_1, \cdots, f_4\}$, which gives a very simple proof of primal infeasibility.

$$-1 = a_0 q(x) + a_1 (x_1 + x_2)^2 \quad \text{with} \quad a_0 = \frac{-2}{2t+1} > 0, \quad a_1 = \frac{-1}{2t+1} > 0$$

In the cone $SOS$.
An Algebraic Dual Problem

Feasibility Problem:
\[ \exists x \in \mathbb{R}^n \text{ s.t. } f_i(x) \geq 0 \text{ (polynomials) } i = 1, \ldots, m \]

“Dual”

Dual Feasibility Problem:
\[ ? \quad -1 \in \text{cone}\{f_1, \ldots, f_m\} \]

If the dual problem is feasible, then the primal problem is infeasible.

In fact, a result called the **Positivstellensatz (P-satz)** implies that **strong duality** holds here.

The above algebraic procedure is searching over **conic combinations**.
An Algebraic Dual Problem

[Ex.] Linear inequalities

Feasibility Problem: 
\[ \exists x \in \mathbb{R}^n \text{ s.t. } Ax \geq 0 \quad c^T x \leq -1 \]

Let us define \( f_i(x) = a_i^T x \) for \( i = 1, \cdots, m \) and \( f_{m+1}(x) = -1 - c^T x \)

Searching over the function \( q(x) = \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_{m+1}(x) \)

Dual Feasibility Problem: 
\[ \exists \lambda_i \geq 0, \mu \geq 0 \text{ s.t. } q(x) = -1 \quad \forall x \in \mathbb{R}^n \]

The above dual condition is \( \lambda^T Ax + \mu(-1 - c^T x) = -1 \quad \forall x \in \mathbb{R}^n \)
which holds iff \( A^T \lambda = \mu c \) and \( \mu = 1 \)

Frakas lemma

If \( \exists \lambda \in \mathbb{R}^m \text{ s.t. } A^T \lambda = c \land \lambda \geq 0 \)

Then there does not exist \( x \in \mathbb{R}^n \text{ s.t. } Ax \geq 0 \land c^T x \leq -1 \)
An Algebraic Dual Problem

[Ex.]

Feasibility Problem:
\[ S = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \geq 0, g(x, y) \geq 0\} \]
where \[ f(x, y) = x - y^2 + 3 \]
\[ g(x, y) = -y - x^2 - 2 \]

By the P-satz, the primal is infeasible iff there exist polynomials \[ s_0, s_1, s_2, s_3 \in \Sigma \] such that
\[ -1 = s_0 + s_1 f + s_2 g + s_3 f g \]

A certificate is given by
\[ s_0 = \frac{1}{3} + 2 \left( y + \frac{3}{2} \right)^2 + 6 \left( x - \frac{1}{6} \right)^2, \quad s_1 = 2, \quad s_2 = 6, \quad s_3 = 0 \]
Optimization Problem

**Optimization Problem:**

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \quad \text{(polynomials)} \\
\text{subject to} & \quad f_i(x) \geq 0 \quad \text{(polynomials)} \quad i = 1, \cdots, m
\end{align*}
\]

**Corresponding Feasibility Problem:**

\[\exists x \in \mathbb{R}^n \text{ s.t. } t - f_0(x) \geq 0 \]
\[f_i(x) \geq 0 \quad i = 1, \cdots, m\]

**Optimization Problem’:**

\[
\begin{align*}
\text{maximize} & \quad t \\
\text{subject to} & \quad t - f_0(x) + \sum_{\{i\}} s_i(x) f_i(x) + \sum_{\{i,j\}} s_{ij}(x) f_i(x) f_j(x) \leq 0
\end{align*}
\]

where \( s_\alpha \in \mathbb{R}[x] \) are SOS

The variables here are (coefficients of) the polynomials \( s_\alpha \in \mathbb{R}[x] \)

We will see later how to approach this kind of problem using SDP
Feasibility of Semi-algebraic Set

Suppose $S$ is a semi-algebraic set represented by polynomial inequalities and equations

$$S = \left\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0, \, \forall i = 1, \cdots, m \right\}$$

Feasibility Problem: $S \neq \emptyset$?

[Non-trivial result] the feasibility problem is decidable
But NP-hard (even for a single polynomial, as we have seen)
We would like to certify infeasibility

The positivstellensatz (P-satz)

Gilbert Stengle

$$S = \emptyset \iff -1 \in \text{cone}\{f_1, \cdots, f_m\} + \text{ideal}\{h_1, \cdots, h_p\}$$

To prove infeasibility, find $f \in \text{cone}\{f_i\}$, $h \in \text{ideal}\{h_i\}$ such that $f + h = -1$

Feasibility of Semi-algebraic Set

[Ex.]

Feasibility Problem:
\[ S = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \geq 0, h(x, y) = 0\} \]
where \[ f(x, y) = x - y^2 + 3 \]
\[ h(x, y) = y + x^2 + 2 \]

By the P-satz, the primal is infeasible iff there exist polynomials \( s_1, s_2 \in \Sigma \) such that
\[ -1 = s_1 + s_2 f + th \]

A certificate is given by
\[ s_1 = \frac{1}{3} + 2 \left( y + \frac{3}{2} \right)^2 + 6 \left( x - \frac{1}{6} \right)^2, \quad s_2 = 2, \quad t = -6 \]
**Feasibility of Semi-algebraic Set**

[Ex.] Farkas Lemma

<table>
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<tr>
<th>Primal Feasibility Problem:</th>
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<tbody>
<tr>
<td>$\exists x \in \mathbb{R}^n$ s.t. $Ax + b \geq 0$ $Cx + d = 0$</td>
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Let $f_i(x) = a_i^T x + b_i$, $h_i(x) = c_i^T x + d_i$

Then this system is infeasible iff

$$-1 \in \text{cone}\{f_1, \ldots, f_m\} + \text{ideal}\{h_1, \ldots, h_p\}$$

Searching over linear combinations, the primal is infeasible if there exist $\lambda \geq 0$ and $\mu$ such that

$$\lambda^T (Ax + b) + \mu^T (Cx + d) = -1$$

Equating coefficients, this is equivalent to

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<tr>
<td>$\exists \lambda, \mu$ s.t. $\lambda^T A + \mu^T C = 0$ $\lambda^T b + \mu^T d = -1$ $\lambda \geq 0$</td>
</tr>
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$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$
Relaxation scheme

Directly provides **hierarchies** of bounds for optimization


Many related open questions:

What sets have “nice” SDP representations?

Links to “rigid convexity” and hyperbolic polynomials

However, they are a *very special* kind of SDP, with very rich algebraic and combinatorial properties
Exploiting this structure is *crucial* in applications.
Semi-algebraic games

Game with an \textit{infinite} number of pure strategies.

In particular, strategy sets are semi-algebraic, defined by polynomial equations and inequalities.

Simplest case:
two players, zero-sum, payoff given by $P(x, y)$, strategy space is a product of intervals.

\textbf{Theorem:}

The value of the game, and the corresponding optimal mixed strategies, can be computed by solving a single SOS program

Perfect generalization of the classical LP for finite games

Related results for multiplayer games and correlated equilibria
SOS Decomposition

[Ex.] \( F(x, y) = 2x^4 + 5y^4 - x^2y^2 + 2x^3y \)

\[
= \begin{bmatrix}
  x^2 \\
  y^2 \\
  xy 
\end{bmatrix}^T \begin{bmatrix}
  q_{11} & q_{12} & q_{13} \\
  q_{12} & q_{22} & q_{23} \\
  q_{13} & q_{23} & q_{33} 
\end{bmatrix} \begin{bmatrix}
  x^2 \\
  y^2 \\
  xy 
\end{bmatrix}
\]

\[
= q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3
\]

\( F \) is SOS iff \( \exists Q \) satisfying the SDP

\[
Q \succeq 0 \quad q_{11} = 2 \quad q_{22} = 5 \quad q_{33} + 2q_{12} = -1
\]

\[
2q_{13} = 2 \quad 2q_{23} = 0
\]

An SDP with equality constraints. Solving, we obtain:

\[
Q = \begin{bmatrix}
  2 & -3 & 1 \\
  -3 & 5 & 0 \\
  1 & 0 & 5
\end{bmatrix} = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix}
  2 & -3 & 1 \\
  0 & 1 & 3
\end{bmatrix}
\]

and therefore \( F(x, y) = \frac{1}{2}(2x^2 - 3y^2 + xy)^2 + \frac{1}{2}(y^2 + 3xy)^2 \)
Polynomials in one variable

If \( f \in \mathbb{R}[x] \), then \( f \) is SOS iff \( f \) is PSD

Every PSD scalar polynomial is the sum of one or two squares

[Ex.] \( f(x) = x^6 - 10x^5 + 51x^4 - 166x^3 + 342x^2 - 400x + 200 \)
\[= (x - 2)^2(x - (2 \pm i))(x - (1 \pm 3i)) \]
\[= (x - 2)^2((x^2 - 3x - 1) \pm i(4x - 7)) \]
\[= (x - 2)^2((x^2 - 3x - 1)^2 + (4x - 7)^2) \]

[All real roots must have even multiplicity and highest coefficient is positive]

Quadratic Polynomials

A quadratic polynomial \( f \) in \( n \) variables is PSD iff \( f \) is SOS

\[
\begin{align*}
\text{f is PSD iff} & \quad f(x) = x^T Q x \quad \text{where} \quad Q \succeq 0 \\
\text{f is SOS iff} & \quad f(x) = \sum_i (v_i^T x)^2 = x^T \left( \sum_i v_i v_i^T \right) x =: x^T Q x 
\end{align*}
\]
The Motzkin Polynomial

A positive semidefinite polynomial, that is not a sum of squares

\[ M(x, y) = x^2 y^4 + x^4 y^2 + 1 - 3x^2 y^2 \]

Nonnegativity follows from the arithmetic-geometric inequality applied to \((x^2 y^4, x^4 y^2, 1)\)

Introduce a nonnegative factor \(x^2 + y^2 + 1\)

Solving the SDPs we obtain the decomposition:

\[
(x^2 + y^2 + 1)M(x, y) = (x^2 y - y)^2 + (xy^2 - x)^2 + (x^2 y^2 - 1)^2 + \frac{1}{4}(xy^3 - x^3 y)^2 + \frac{3}{4}(xy^3 + x^3 y - 2xy)^2
\]
The Univariate Case

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2d} x^{2d} \]

\[
\begin{bmatrix}
1 \\
x \\
\vdots \\
x^d
\end{bmatrix}^T
\begin{bmatrix}
q_{00} & q_{01} & \cdots & q_{0d} \\
q_{01} & q_{11} & \cdots & q_{1d} \\
\vdots & \vdots & \ddots & \vdots \\
q_{0d} & q_{1d} & \cdots & q_{dd}
\end{bmatrix}
\begin{bmatrix}
1 \\
x \\
\vdots \\
x^d
\end{bmatrix}
\]

\[
= \sum_{i=0}^{d} \left( \sum_{j+k=i} q_{jk} \right) x^i
\]

In the univariate case,
the SOS condition is exactly equivalent to nonnegativity

The matrices \( A_i \) in the SDP have a Hankel structure.
This can be exploited for efficient computation.
Necessary Conditions

Suppose $f = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$; then

\[ f \text{ is PSD} \Rightarrow d \text{ is even, } c_d > 0 \text{ and } c_0 \geq 0 \]

What is the analogue in $n$ variables?

[Ex.] The Newton polytope

Suppose $f = \sum_{\alpha \in M} c_\alpha x^\alpha$

The set of monomials $M \subset \mathbb{N}^n$ is called the frame of $f$

The Newton polytope of $f$ is its convex hull

\[ \text{new}(f) = \text{co}(\text{frame}(f)) \]

The example shows

\[ f = 7x^4y + x^3y + x^2y^4 + x^2 + 3xy \]
Necessary Conditions for nonnegativity

If \( f \in \mathbb{R}[x_1, \cdots, x_n] \) is PSD, then every vertex of \( \text{new}(f) \) has even coordinates, and a positive coefficient.

[Ex.] \( f = 7x^4y + x^3y + x^2y^4 + x^2 + 3xy \)
is not PSD, since term \( 3xy \) has coords \((1, 1)\).

[Ex.] \( f = 7x^4y + x^3y - x^2y^4 + x^2 + 3y^2 \)
is not PSD, since term \(-x^2y^4\) has a negative coefficient.

Properties of Newton Polytopes

Products: \( \text{new}(fg) = \text{new}(f) + \text{new}(g) \)

Consequently, \( \text{new}(f^n) = n\text{new}(f) \)

If \( f \) and \( g \) are PSD polynomials then
\[
f(x) \leq g(x) \quad \forall x \in \mathbb{R}^n \quad \Rightarrow \quad \text{new}(f) \subseteq \text{new}(g)
\]

\[
f = \sum_{i=1}^{t} g_i^2 \quad \Rightarrow \quad \text{new}(g_i) \subseteq \frac{1}{2}\text{new}(f)
\]

SOS decomposition
Sparse SOS Decomposition

[Ex.] Find an SOS representation for

\[ f = 4x^4y^6 + x^2 - xy^2 + y^2 \]

The squares in an SOS decomposition can only contain the monomials

\[ \frac{1}{2} \text{new}(f) \cap \mathbb{N}^n = \{ x^2y^3, xy^2, xy, x, y \} \]

Without using sparsity, we would include all 21 monomials of degree less than 5 in the SDP

With sparsity, we only need 5 monomials
SOS Programming: SPOT

SPOT: Systems Polynomial Optimization Tools

Alexandre Megretski

SOS Problems: Extensions

Other linear (partially) differential inequalities

1. Lyapunov: \[ V(x) \leq 0, \left( \frac{\partial V}{\partial x} \right)^T f(x) \leq 0, \ \forall x \]

2. Hamilton-Jacobi: \[ V(x, t) \leq 0, -\frac{\partial V}{\partial t} + H \left( x, \frac{\partial V}{\partial x} \right) \leq 0, \ \forall (x, u, t) \]

Many possible variations:

- Nonlinear $H_\infty$ analysis, parameter dependent Lyapunov functions,
- Constrained nonlinear systems, systems with time-delay, hybrid systems
- Mixed integer programming (MIP) problem etc.

Can also do local results (for instance, on compact domains)

Polynomial and rational vector fields, or functions with an underlying algebraic structure

Natural extension of the LMIs for the linear case

Only for analysis, proper synthesis is a trick problem
Valid inequalities and Cones

The function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is called a **valid inequality** if
\[
f(x) \geq 0 \quad \text{for all feasible } x
\]

Given a set of inequality constraints, we can generate others as follows:

1. If \( f_1 \) and \( f_2 \) define valid inequalities, then so does \( h(x) = f_1(x) + f_2(x) \)
2. If \( f_1 \) and \( f_2 \) define valid inequalities, then so does \( h(x) = f_1(x)f_2(x) \)
3. For any \( f \), the function \( h(x) = f(x)^2 \) defines a valid inequality

\( \mathbb{R}[x_1, \cdots, x_n] : \) the set of polynomial functions on \( \mathbb{R} \) with real coefficients

A set of polynomials \( P \subset \mathbb{R}[x_1, \cdots, x_n] \) is called a **cone** if

1. \( f_1 \in P \) and \( f_2 \in P \) implies \( f_1 + f_2 \in P \)
2. \( f_1 \in P \) and \( f_2 \in P \) implies \( f_1f_2 \in P \)
3. \( f \in \mathbb{R}[x_1, \cdots, x_n] \) implies \( f^2 \in P \)

It is called a **proper cone** if \(-1 \not\in P\)

By applying the above rules to the inequality constraint functions (**algebra**), we can generate a cone of valid inequalities
Algebra: Cones

For $S \subset \mathbb{R}^n$, the cone defined by $S$ is
\[ \mathcal{C}(S) = \{ f \in \mathbb{R}[x_1, \ldots, x_n] \mid f(x) \geq 0 \ \forall x \in S \} \]

If $P_1$ and $P_2$ are cones, then so is $P_1 \cap P_2$

Every cone contains the set of SOS polynomials $\Sigma$, which is the smallest cone

The set $\text{monoid}\{f_1, \ldots, f_m\} \subset \mathbb{R}[x_1, \ldots, x_n]$ is the set of all finite products of polynomials $f_i$, together with 1

The smallest cone containing the polynomials $f_1, \ldots, f_m$ is

\[ \text{cone}\{f_1, \ldots, f_m\} = \{ \sum_{i=1}^{r} s_i g_i \mid s_0, \ldots, s_r \in \Sigma, g_i \in \text{monoid}\{f_1, \ldots, f_m\} \} \]

If $f_1, \ldots, f_m$ are valid inequalities, then so is every polynomial in $\text{cone}\{f_i\}$

The polynomial $h$ is an element of $\text{cone}\{f_1, \ldots, f_m\}$ iff

\[ h(x) = s_0 + \sum_{\{i\}} s_i g_i + \sum_{\{i,j\}} s_{ij} g_i g_j + \sum_{\{i,j,k\}} s_{ijk} g_i g_j g_k + \cdots \]

where a coefficient $s_\alpha \in \mathbb{R}[x]$ that is a sum of squares

(linear combination of squarefree products of $f_i$ with SOS coefficients)
An algebraic Dual Problem: Interpretation

Searching the Core

Lagrange duality is searching over linear combinations with nonnegative coefficients to fine a globally negative function as a certificate

\[ h(x) = \lambda_1 f_1 + \cdots + \lambda_m f_m \]

The algebraic procedure is searching over conic combinations

\[ h(x) = s_0 + \sum_{i} s_i g_i + \sum_{i,j} s_{ij} g_i g_j + \sum_{i,j,k} s_{ijk} g_i g_j g_k + \cdots \]

where a coefficient \( s_\alpha \in \mathbb{R}[x] \) that is a sum of squares

Formal Proof

View \( f_1, \cdots, f_m \) are predicates, with \( f_i(x) \geq 0 \) meaning that \( x \) satisfies \( f_i \)

Then \( \text{cone}\{f_1, \cdots, f_m\} \) consists of predicates which are logical consequences of \( f_1, \cdots, f_m \)

If we find -1 in the cone, then we have a proof by contradiction

The objective is to automatically search the cone for negative functions: i.e., proofs of infeasibility
Frakas Lemma

(Algebraic definition)

Frakas lemma states that the following are strong alternatives:

(i) \( \exists \lambda \in \mathbb{R}^m \text{ s.t. } A^T \lambda = c \land \lambda \geq 0 \)

(ii) \( \exists x \in \mathbb{R}^n \text{ s.t. } Ax \geq 0 \land c^T x < 0 \)

(Geometric interpretation) (\( \therefore \) Lagrangian duality)

(i) \( c \) is in the convex cone
\[ \{ A^T \lambda \mid \lambda \geq 0 \} \]

(ii) \( x \) defines the hyperplane
\[ \{ y \in \mathbb{R}^n \mid y^T x = 0 \} \]
with separates \( c \) from the cone
Valid Equality constraints and Ideals

The function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a valid equality constraint if
\[ f(x) = 0 \quad \text{for all feasible } x \]

Given a set of equality constraints, we can generate others as follows:

1. If $f_1$ and $f_2$ are valid equalities, then so is $f_1 + f_2$
2. For any $h \in \mathbb{R}[x_1, \cdots, x_n]$, if $f$ is a valid equality, then so is $hf$

Using these will make the dual bound tighter

A set of polynomials $I \subset \mathbb{R}[x_1, \cdots, x_n]$ is called a ideal if

1. $f_1 + f_2 \in I$ for all $f_1 \in I$, $f_2 \in I$
2. $hf \in I$ for all $f \in I$ and $h \in \mathbb{R}[x_1, \cdots, x_n]$

Given $f_1, \cdots, f_m$, we can generate an ideal of valid equalities by repeatedly applying these rules
Algebra: Ideals

This gives the ideal generated by \( f_1, \cdots, f_m \),

\[
\text{ideal}\{f_1, \cdots, f_m\} = \left\{ \sum_{i=1}^{m} h_i f_i \mid h_i \in \mathbb{R}[x_1, \cdots, x_n] \right\}
\]

Generator of an ideal

Every polynomial in \( \text{ideal}\{f_1, \cdots, f_m\} \) is a valid equality

\( \text{ideal}\{f_1, \cdots, f_m\} \) is the smallest ideal containing \( f_1, \cdots, f_m \)

The polynomials \( f_1, \cdots, f_m \) are called the generators/basis of the ideal

Properties of ideals

If \( I_1 \) and \( I_2 \) are ideals, then so is \( I_1 \cap I_2 \)

An ideal generated by one polynomial is called a principal ideal
Algebra: The real Nullstellensatz (N-satz)

\( \Sigma \) : the cone of polynomials representable as SOS

Suppose \( h_1, \ldots, h_m \in \mathbb{R}[x_1, \ldots, x_n] \)

\[
-1 \in \Sigma + \text{ideal}\{h_1, \ldots, h_m\} \iff \mathcal{V}_\mathbb{R}\{h_1, \ldots, h_m\} = \emptyset
\]

Equivalently, there is no \( x \in \mathbb{R}^n \) such that

\[
h_i(x) = 0 \ \forall i = 1, \ldots, m
\]

iff there exists \( t_1, \ldots, t_m \in \mathbb{R}[x_1, \ldots, x_n] \) and \( s \in \Sigma \) such that

\[
-1 = s + t_1h_1 + \cdots + t_mh_m
\]

[Ex.] Suppose \( h(x) = x^2 + 1 \) Then clearly \( \mathcal{V}_\mathbb{R}\{h\} = \emptyset \)

We saw earlier that the complex N-satz cannot be used to prove

But we have \( -1 = s + th \) with \( s(x) = x^2 \) and \( t(x) = -1 \)

and so the real N-satz implies \( \mathcal{V}_\mathbb{R}\{h\} = \emptyset \)

The polynomial equation \( -1 = s + th \) gives a certificate of infeasibility
Algebra: The Positivstellensatz (P-satz)

Feasibility Problem for basic semi-algebraic sets:
\[ \exists x \in \mathbb{R}^n \quad \text{s.t.} \quad f_i(x) \geq 0 \quad i = 1, \ldots, m \]
\[ h_j(x) = 0 \quad j = 1, \ldots, p \]

Call the feasible set \( S \); recall

Every polynomial in \( \text{cone}\{f_1, \ldots, f_m\} \) is nonnegative on \( S \)

Every polynomial in \( \text{ideal}\{h_1, \ldots, h_p\} \) is zero on \( S \)

\[ S = \emptyset \iff -1 \in \text{cone}\{f_1, \ldots, f_m\} + \text{ideal}\{h_1, \ldots, h_p\} \]

Centerpiece of real algebraic geometry

Dual Feasibility Problem:
\[ \exists t_i \in \mathbb{R}[x_1, \ldots, x_n] \text{ and } \exists s_\alpha \in \Sigma \quad \text{s.t.} \]
\[ -1 = \sum_{\{i\}} t_i h_i + s_0 + \sum_{\{i\}} s_i f_i + \sum_{\{i,j\}} s_{ij} f_i f_j + \cdots \]

These are strong alternatives
Testing the P-satz

Dual Feasibility Problem:
\[ \exists t_i \in \mathbb{R}[x_1, \ldots, x_n] \text{ and } \exists s_\alpha \in \Sigma \text{ s.t. } \]
\[ -1 = \sum_{\{i\}} t_i h_i + s_0 + \sum_{\{i\}} s_i f_i + \sum_{\{i,j\}} s_{ij} f_i f_j + \cdots \]

This is a convex feasibility problem in \( t_i, s_\alpha \)

To solve it, we need to choose a subset of the cone to search; i.e., the maximum degree of the above polynomial; then the problem is a SDP

This gives a hierarchy of syntactically verifiable certificates

The validity of a certificate may be easily checked; e.g., linear algebra, random sampling

Unless \( \text{NP}=\text{co-NP} \), the certificates cannot always be polynomially sized
Infeasibility Certificates
and associated computational techniques

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Common generalization of Hilbert’s N-satz and LP duality
Guarantees the existence of **infeasibility certificates** for real solutions of systems of polynomial equations
Infeasibility Certificates: Some Theorems

(Range/Kernel)

\[ Ax = b \text{ is infeasible} \]
\[ \implies \]
\[ \exists \mu \text{ s.t. } A^T \mu = 0, \ b^T \mu = -1 \]

(Hilbert’s N-satz)

Let \( f_i(z), i = 1, \cdots, m \) be polynomials in \( z \in \mathbb{C}^n \). Then,

\[ f_i(z) = 0, \ i = 1, \cdots, m \]

is infeasible in \( \mathbb{C}^n \)

\[ \implies \]
\[-1 \in \text{ideal}(f_1, \cdots, f_m)\]

(Farkas Lemma)

\[ Ax + b = 0 \text{ is infeasible} \]
\[ Cx + d \geq 0 \]
\[ \implies \]
\[ \exists \lambda \geq 0, \ \mu \text{ s.t. } A^T \mu + C^T \lambda = 0 \]
\[ b^T \mu + d^T \lambda = -1 \]

(P-satz)

\[ f_i(z) = 0, \ i = 1, \cdots, m \]
\[ g_i(z) \geq 0, \ i = 1, \cdots, p \]

is infeasible in \( \mathbb{R}^n \)

\[ \implies \]
\[ \exists F(x), G(x) \in \mathbb{R}[x] \text{ s.t.} \]
\[ F(x) + G(x) = -1 \]
\[ F(x) \in \text{ideal}(f_1, \cdots, f_m) \]
\[ G(x) \in \text{cone}(g_1, \cdots, g_p) \]
Liftings

By going to higher dimensional representations, things may become easier:

“complicated” sets can be the projection of much simpler ones

A polyhedron in $\mathbb{R}^n$ with a “small” number of faces can project to a lower dimensional space with exponentially many faces

Basic semialgebraic sets can project into non-basic semialgebraic sets

An essential technique in integer programming

[Ex.] \[
\begin{align*}
\text{minimize} & \quad (x - 3)^2 & \text{subject to} & \quad x(x - 4) \geq 0 \\
\end{align*}
\]

Feasible set $[-\infty, 0] \cup [4, \infty]$ Not convex

Lifting $L(x) = (x, x^2) =: (x, y)$

\[
\begin{align*}
\text{minimize} & \quad y - 6x + 9 \\
\text{subject to} & \quad y - 4x \geq 0 \quad \text{and} \quad \begin{bmatrix} 1 & x \\ x & y \end{bmatrix} \succeq 0
\end{align*}
\]
The dual side of SOS: Moment sequences

The SDP dual of the SOS construction gives efficient semidefinite liftings

[Ex.] For the univariate case

\[ L : \mathbb{R} \rightarrow \mathbb{S}^{d+1} \text{ with } L(x) = \begin{bmatrix} 1 & x & \cdots & x^d \\ x & x^2 & \cdots & x^{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ x^d & x^{d+1} & \cdots & x^{2d} \end{bmatrix} \]

The matrices \( L(x) \) are Hankel, positive semidefinite, and rank one.

The convex hull \( \text{conv} L(x) \) contains only PSD Hankel matrices

\[ \text{Hankel}(w) := \begin{bmatrix} 1 & w_1 & \cdots & w_d \\ w_1 & w_2 & \cdots & w_{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ w_d & w_{d+1} & \cdots & w_{2d} \end{bmatrix} \]

In fact, in the univariate case every PSD Hankel is in the convex hull
Convex Combination, Convex Hull and Convex Cone

Convex combination of \( x_1, \cdots, x_k \):

Any point \( x \) of the form

\[
x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k
\]

with \( \theta_1 + \cdots + \theta_k = 1, \theta_i \geq 0 \)

Convex hull \( \text{conv} S \):

Set of all convex combinations of points in \( S \)

Conic (nonnegative) combination of \( x_1 \) and \( x_2 \):

Any point \( x \) of the form

\[
x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k
\]

with \( \theta_1 \geq 0, \theta_2 \geq 0 \)

Convex Cone:

Set that contains all conic combinations of points in the set
Positive Semidefinite (PSD) Cone

Notations:

\[ S^n \] : set of symmetric \( n \times n \) matrices

\[ S_+^n = \{ X \in S^n \mid X \succeq 0 \} \]

: positive semidefinite \( n \times n \) matrices (convex cone)

\[ X \in S_+^n \iff z^T X z \geq 0 \ \forall z \]

\[ S_{++}^n = \{ X \in S^n \mid X \succ 0 \} \]

: positive definite \( n \times n \) matrices

[Ex.]

\[
\begin{bmatrix}
  x & y \\
  y & z \\
\end{bmatrix} \in S_+^2
\]

**Algebraic structure**

**Sparseness**: few non-zero coefficients
- Newton polytopes techniques

**Ideal structure**: equality constraints
- SOS on *quotient rings*. Compute in the coordinate ring. Quotient bases.

**Graph structure**:
- Dependency graph among the variables

**Symmetries**: invariance under a group
- SOS on *invariant rings*.
- Representation theory and invariant-theoretic methods

**Numerical structure**

**Rank one SDPs**
- Dual coordinate change makes all constraints rank one
- Efficient computation of Hessians and gradients

**Representations**
- Interpolation representation. Orthogonalization

**Displacement rank**
- Fast solvers for search direction
Algebraic Structure: SOS over everything…

Algebraic tools are *essential* to exploit problem structure:

<table>
<thead>
<tr>
<th><strong>Standard</strong></th>
<th><strong>Equality constraints</strong></th>
<th><strong>Symmetries</strong></th>
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</thead>
<tbody>
<tr>
<td>Polynomial ring $\mathbb{R}[x]$</td>
<td>Quotient ring $\mathbb{R}[x]/I$</td>
<td>Invariant ring $\mathbb{R}[x]^G$</td>
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<tr>
<td>Monomials (deg ≤ $k$)</td>
<td>$\text{Standard monomials}$</td>
<td>Isotypic components</td>
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<td>Hibert series</td>
<td>Molien series</td>
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<td></td>
<td>Finite convergence for zero dimensional ideals</td>
<td>Block diagonalization</td>
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</tbody>
</table>

\[
\frac{1}{(1 - \lambda)^n} = \sum_{k=0}^{\infty} \left( \binom{n + k - 1}{k} \right) \lambda^k
\]