Robust Control

Spring, 2019

Instructor: Prof. Masayuki Fujita (S5-303B)

6th class

Tue., 21st May, 2019, 10:45～12:15,

S423 Lecture Room

6. Design Example 1

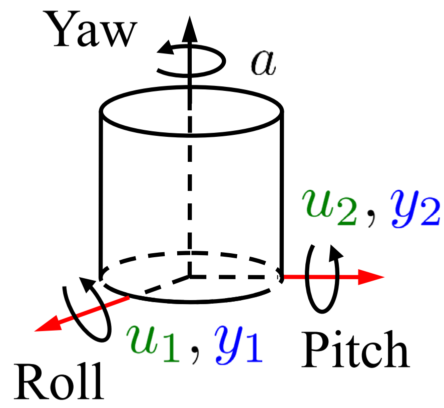
6.1 Spinning Satellite: H_∞ Control [SP05, Sec. 3.7]

6.2 2nd Report

Reference:

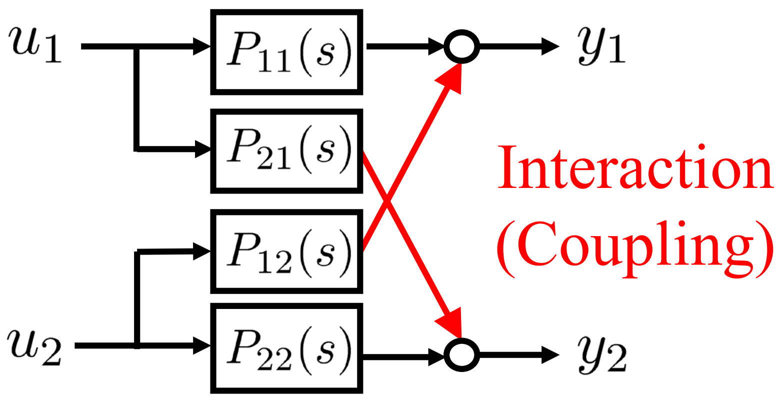
[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.

Spinning Satellite: Nominal Plant Model



Transfer Function Matrix

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$



MATLAB Command

```
N = { [1 -100],[10 10];[-10 -10],[1 -100] } ;
D = [1 0 100] ;
Pnom = tf(N,D) ;
Pnom = ss(Pnom,'min') ;
```

State Space Representation

$$P(s) = C(sI - A)^{-1}B + D$$

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$$A = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 10 \\ -10 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

MATLAB Command

```
sysA = [0 10; -10 0]; sysB = eye(2);
sysC = [1 10; -10 1]; sysD = zeros(2);
Pnom = ss(sysA, sysB, sysC, sysD);
```

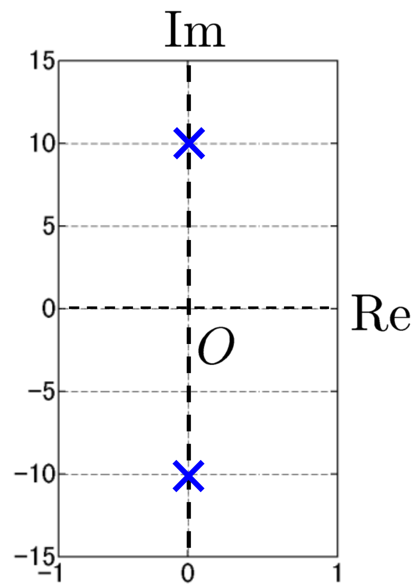
Spinning Satellite: Characteristics of Nominal Plant Model

Poles (Stability)

$p = \pm 10j$ (at Imaginary axis)

Unstable Poles

Vibrational System



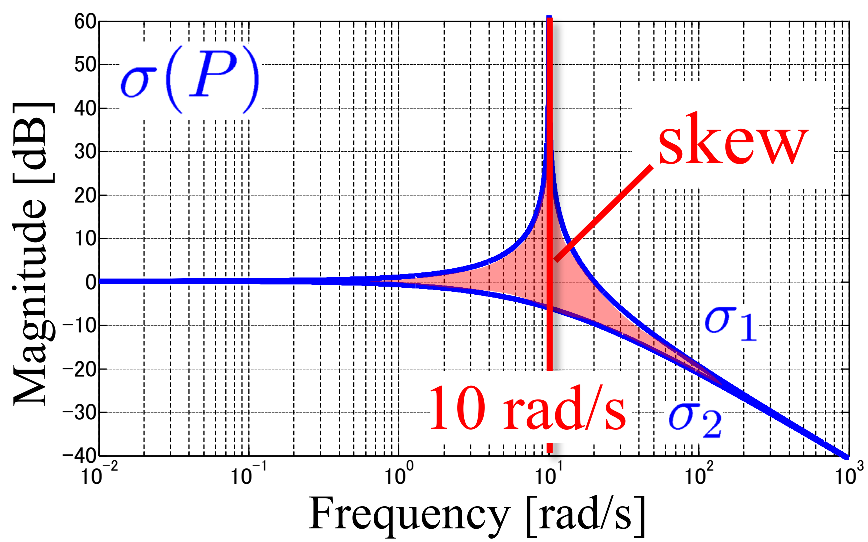
MATLAB Command
pole(Pnom) tzero(Pnom)

MATLAB Command
figure pzmap(Pnom)

Multivariable Zeros

None

Frequency Response σ-plot



MATLAB Command
sigma(Pnom)

Condition Number

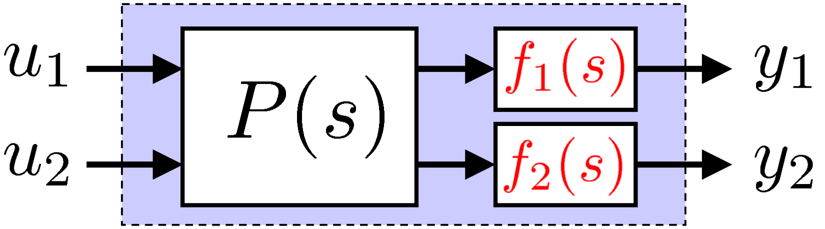
$$\gamma(P) := \frac{\bar{\sigma}(P)}{\underline{\sigma}(P)} \quad \text{[SP05, p. 82]}$$

Operating Frequency Range

Spinning Satellite: Plant Model with Output Uncertainty

Uncertain Plant Model (Real System)

$$\tilde{P}(s) = \begin{bmatrix} f_1(s) & 0 \\ 0 & f_2(s) \end{bmatrix} P(s)$$



$$f_i(s) = k_i \frac{-\frac{\theta_i}{2}s + 1}{\frac{\theta_i}{2}s + 1}, \quad i = 1, 2$$

Gain Margin: $0.8 \leq k_i \leq 1.2$

Delay Margin: $0 \leq \theta_i \leq 0.02$

(Processing Time: 20ms)

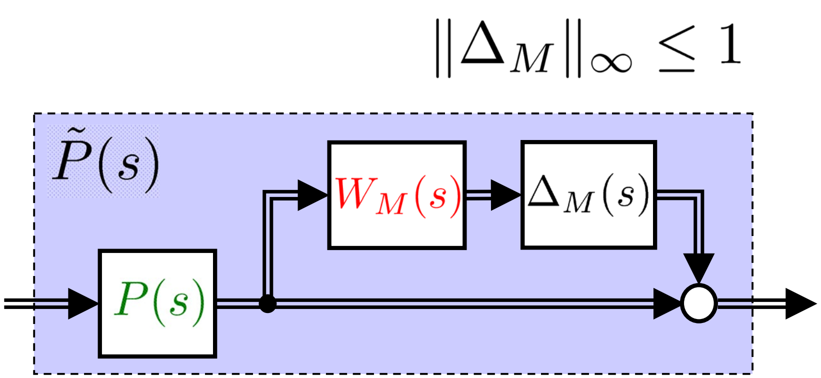
Multiplicative (Output) Uncertainty

$$\Pi_0 = \{\tilde{P}(s) \mid \tilde{P}(s) = (I + \Delta_M(s)W_M(s))P(s), \ \|\Delta_M\|_\infty \leq 1\}$$

Uncertainty Weight:

$$W_M(s) = w_M(s)I_2 \ ,$$

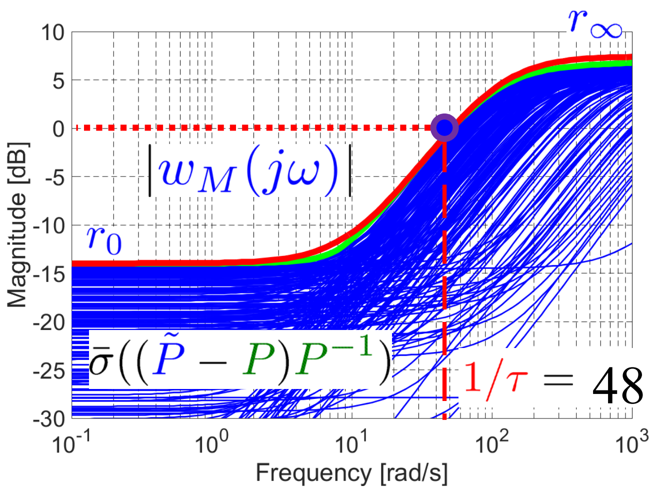
$$w_M(s) = \frac{\tau s + r_0}{\frac{\tau}{r_\infty}s + 1}$$



Spinning Satellite: Uncertainty Weight

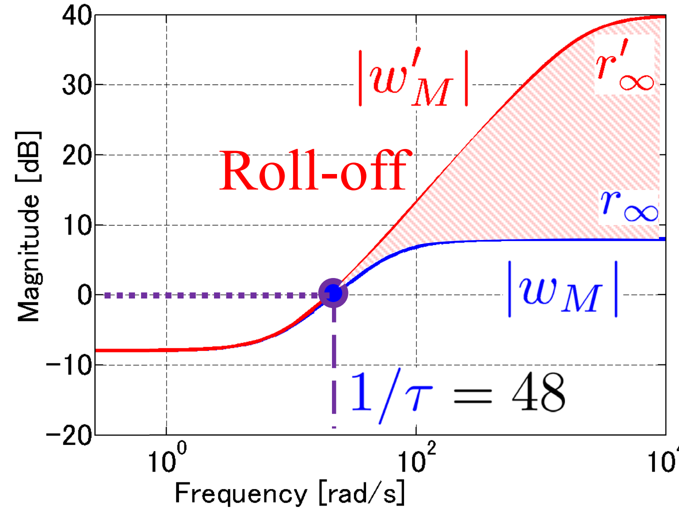
$$w_M(s) = \frac{0.021s + 0.2}{0.0091s + 1}$$

$$\left(\begin{array}{l} \tau = 0.021, \\ 1/\tau = 48 \text{ rad/s} \\ r_0 = 0.2, \\ r_\infty = 2.3 \end{array}\right)$$



MATLAB Command

```
r0 = 0.2; rinf = 2.3; tau = 0.021;
wM = tf([tau r0], [tau/rinf 1]);
WM = eye(2)*wM;
WM = ss(WM) ;
```



Update

Uncertain Factors:

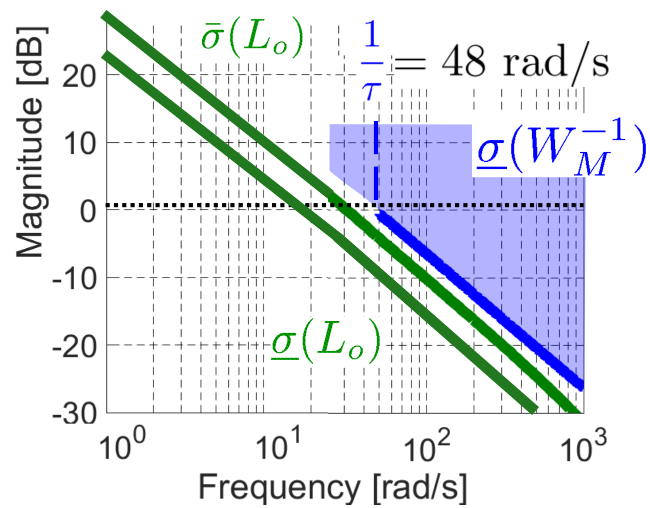
Gain, Time Delay +

Unmodeled Dynamics (High Freq.)

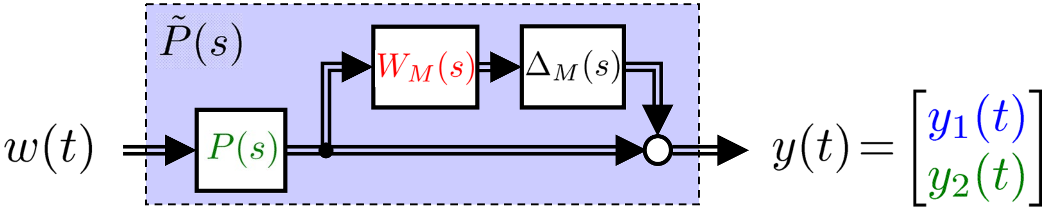
➡ $r_\infty = 2.3 \rightarrow 100$

$$w'_M(s) = \frac{0.021s + 0.2}{2.1 \times 10^{-4}s + 1}$$

Target Loop (Roll-off)

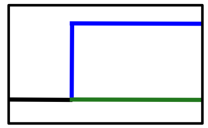


Spinning Satellite: Time Responses for Uncertain Plant Model

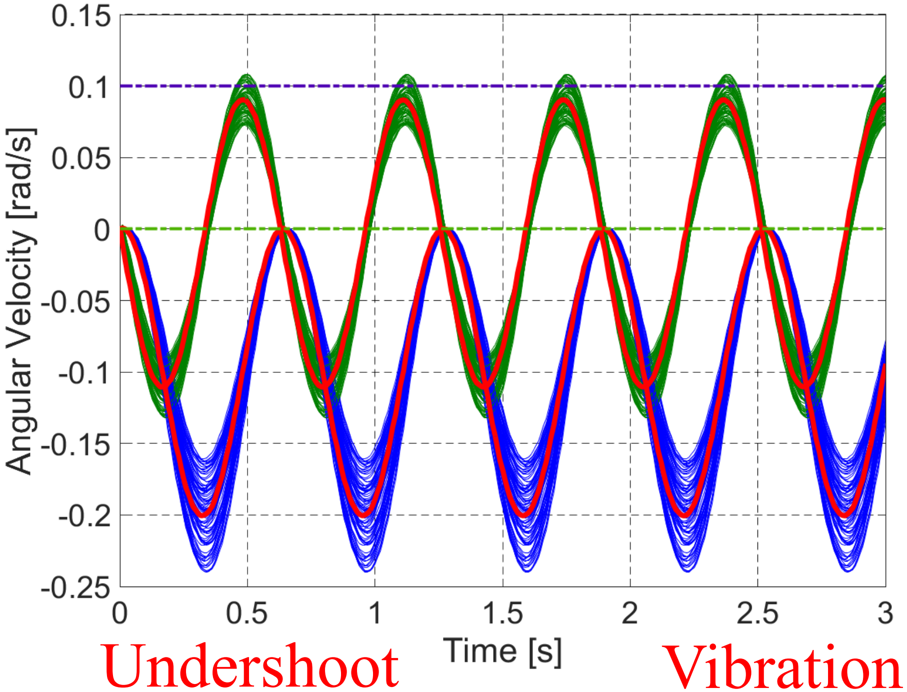


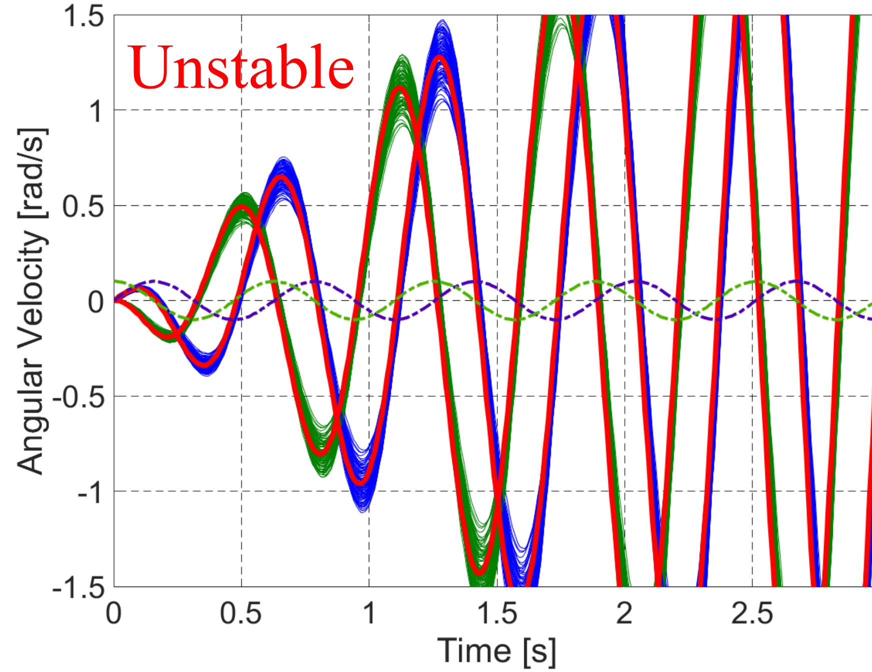
Reference
Nominal Model
Uncertain Model

$$w(t) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$



$$w(t) = \begin{bmatrix} 0.1\sin(\omega t) \\ 0.1\cos(\omega t) \end{bmatrix} \quad \omega = 10 \text{ rad/s}$$





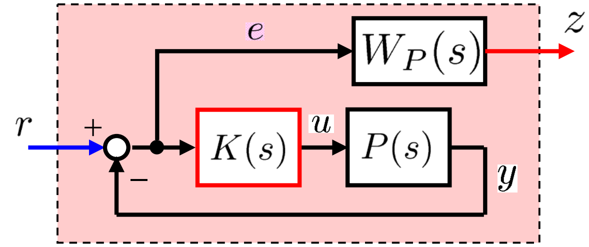
MATLAB Command

```
time = 0:0.01:3;
step_ref = ones(1,length(time));
ref = [0.1*step_ref'; 0*step_ref'];
```

Reference Signal

```
figure
plot(time,ref,'g-.')
```

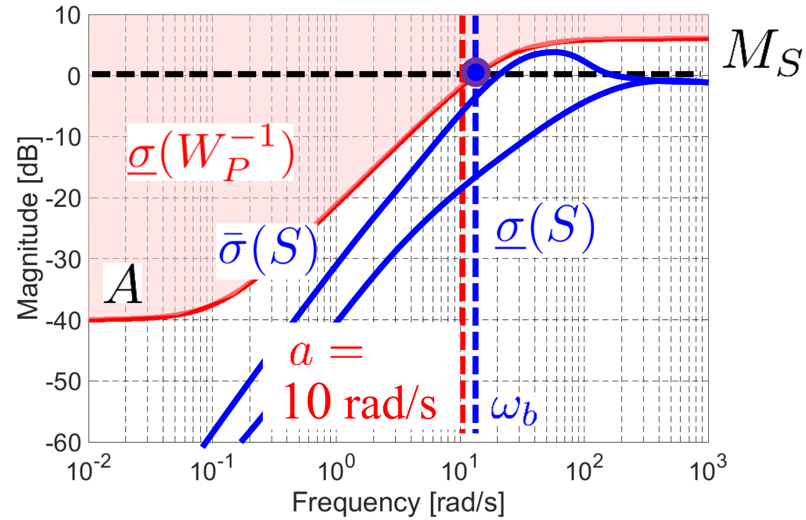
Spinning Satellite: Performance Weight



$$W_P(s) = w_p(s) I_2,$$

$$w_p(s) = \frac{\frac{1}{M_S}s + \omega_b}{s + \omega_b A} = \frac{0.5s + 11.5}{s + 0.115}$$

$$\begin{bmatrix} \omega_b = 11.5\ (\geq 1.15|p|) \\ M_s = 2, A = 0.01 \end{bmatrix}$$



MATLAB Command

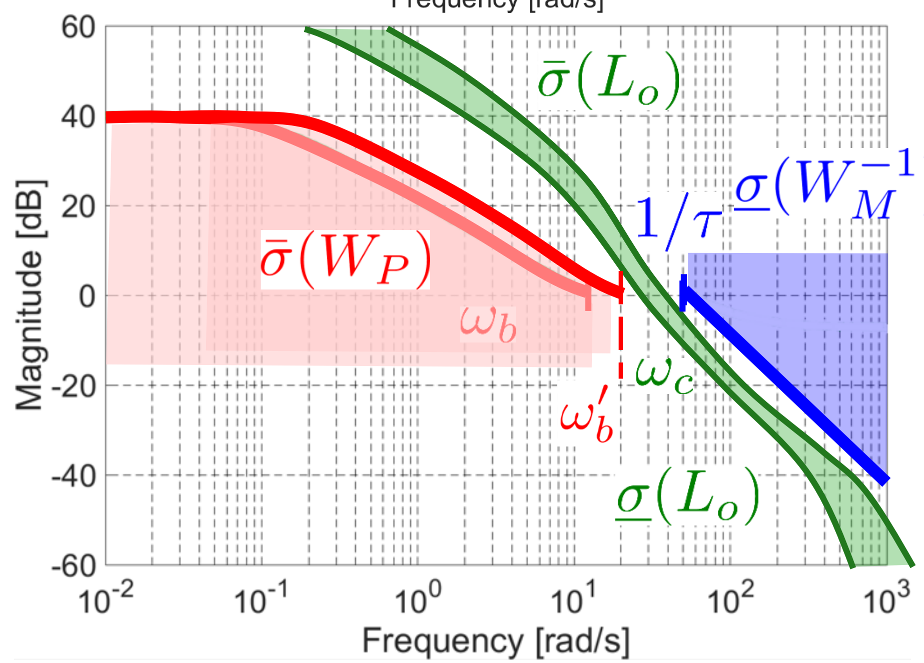
```
Ms = 2; A = 1e-2; wb = 11.5;
wP = tf([1/Ms wb], [1 wb*A]);
WP = eye(2)*wP; WP = ss(WP) ;
```

Update $\omega_b = 11.5 \rightarrow 20$

$M_S = 2 \rightarrow 8$ (Trade-off)

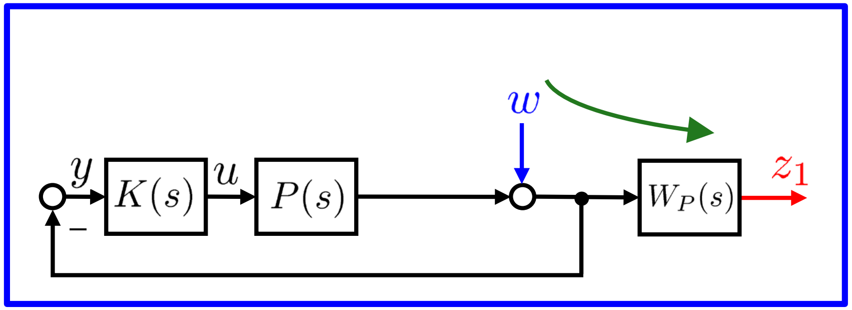
$A = 0.01 > 0$

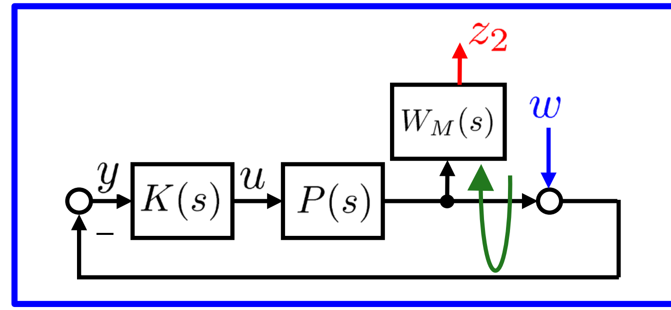
$$w_p'(s) = \frac{0.125s + 20}{s + 0.2}$$

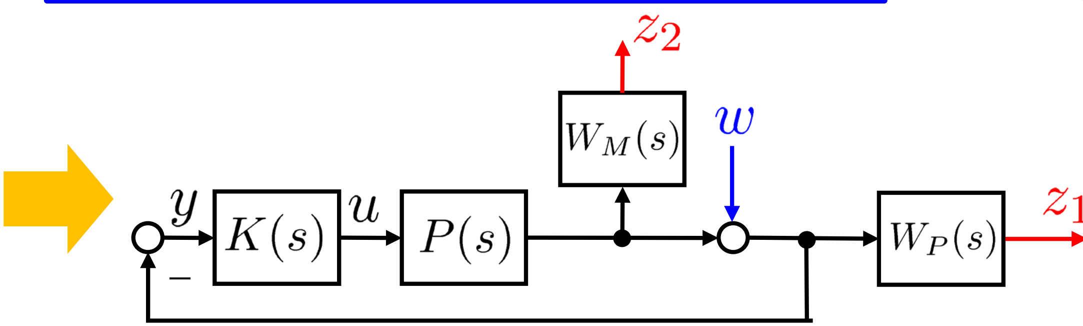


Spinning Satellite: Control Problem Formulation

Nominal Performance $\|W_P(s)S_o(s)\|_\infty$ Robust Stability $\|W_M(s)T_o(s)\|_\infty$





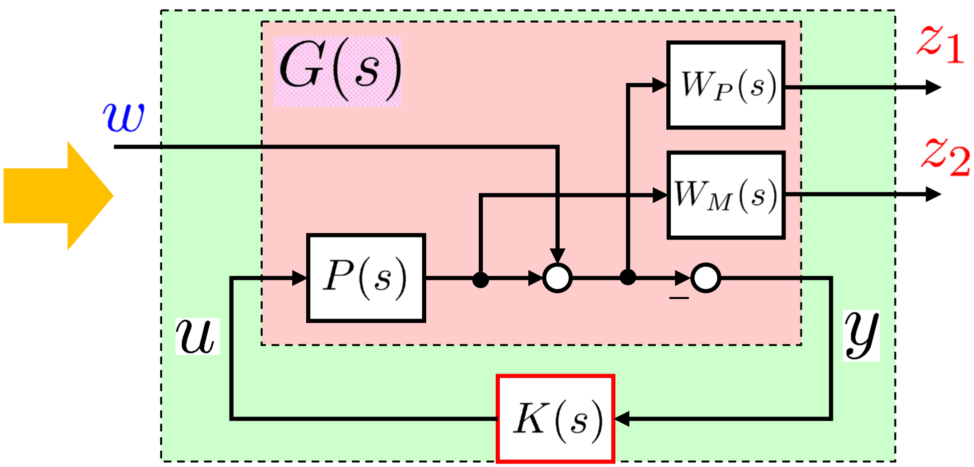


MATLAB Command

```
%Generalized Plant%
systemnames = 'Pnom WP WM';
inputvar = '[w(2);u(2)]';
outputvar = '[WP;WM;-w-Pnom]';
input_to_Pnom= '[u]';
input_to_WP = '[w+Pnom]';
input_to_WM = ' [Pnom]';
G = sysic;
```

Generalized Plant

sysic

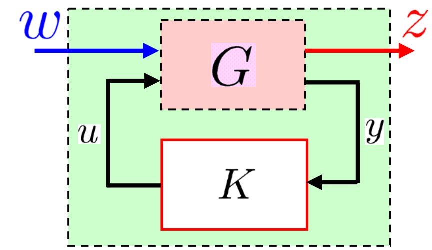


$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_P(s)S_o(s) \\ -W_M(s)T_o(s) \end{bmatrix} w$$

$$= F_l(G, K)$$

Mixed Sensitivity

H_∞ Optimal Controller hinfsyn



$$z = F_l(G,K)w$$

H_∞ Control Problem

Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that $\|F_l(G,K)\|_\infty < \gamma$

```
[K, CL, gam, info] = hinfsyn(G, nmeas, ncon, key1, value1, key2, value2, …)
```

input arguments

- G generalized plant
- nmeas number of measurement outputs
- ncon number of control inputs

output arguments

- K LTI controller
- CL closed loop system which consists of K and G
- gam H_∞ norm of closed loop system
- info information of output results

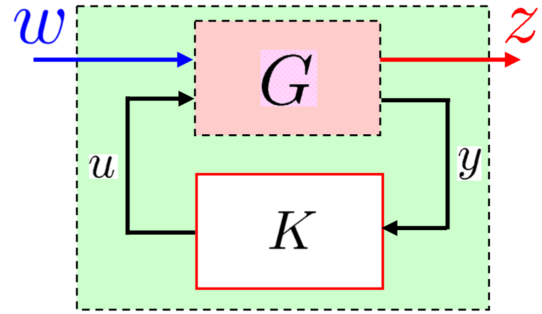
key settings

- Gmax upper limit of gam(=Inf)
- Gmin lower limit of gam(=0)
- Tolgam relative error of gam(=0.01)
- So frequency at which entropy is assessed (default=Inf)
- Display off : not show setting process; on : show setting process
- Method ric : Ricatti solution (default); lmi : LMI solution; maxe : max entropy solution

MATLAB Command

```
nmeas = 2; ncon = 2;
[Khi,CLhi,ghi,hiinfo] = hinfsyn(G,nmeas,ncon);
ghi
Fhi=loopsens(Pnom,Khi);
```

H_∞ Optimal Controller — hinfsyn



Assumptions for Generalized Plant

(1) (A, B_2) :stabilizable, (C_2, A) :detectable

(2) (A, B_1) :controllable, (C_1, A) :observable

→ [Full rank on the imaginary axis]

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

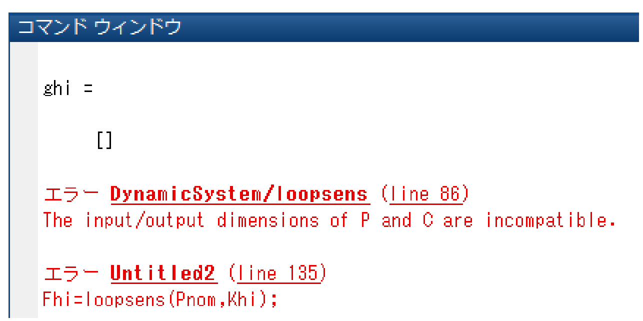
"If the Robust Control toolbox of MATLAB complains, then it probably means that your control problem is not well formulated and you should think again" [SP05, P355]

Infeasible Weights

[Ex.] $w_p(s) = \dfrac{\frac{1}{M_S}s + \omega_b}{s}$ $(A = 0)$

Practically no problem

Error Example


コマンド ウィンドウ

```
ghi =

     []

エラー DynamicSystem/loopsens (line 86)
The input/output dimensions of P and C are incompatible.

エラー Untitled2 (line 135)
Fhi=loopsens(Pnom,Khi);
```

Data Structure

名前	値
CLhi	[ ]
Khi	[ ]
ghi	[ ]
hiinfo	[ ]

Spinning Satellite: γ-iteration to obtain H_∞ Controller [SP05, p. 358]

Find K such that $\|F_l(G,K)\|_\infty < \gamma$

Check 1)

Appropriately sub-optimal

(Default settings)

$\gamma = 0.6719 < 1 \quad (\gamma_{opt} = 0.6650)$

```
Resetting value of Gamma min based on D_11, D_12, D_21 terms

Test bounds:     0.1250 <  gamma  <=     0.6719

gamma    hamx_eig  xinf_eig  hamy_eig   yinf_eig   nrho_xy   p/f
 0.672    1.6e+01   9.2e-02   5.8e-11   -1.8e-21    0.0036   p
 0.398    3.1e-12# ********   5.8e-11   -1.1e-21   ********  f
 0.535    1.1e+01  -1.4e+00#  5.8e-11   -1.5e-23    0.0002   f
 0.604    1.4e+01  -4.1e+00#  5.8e-11   -7.2e-22    0.0004   f
 0.638    1.5e+01  -1.1e+01#  5.8e-11    3.1e-22    0.0009   f
 0.655    1.6e+01  -3.0e+01#  5.8e-11   -2.6e-21    0.0024   f
 0.663    1.6e+01  -1.9e+02#  5.8e-11   -5.5e-21    0.0144   f
Gamma value achieved:     0.6719
```

Check 2) $\gamma \to \infty$ (Gmax $= 100$)

($K \to$ LQG Controller/H_2)

$$\|F_l(G,K)\|_\infty = 1.0232$$

Check 3) $\gamma = 0.5$ (Gmax $= 0.5$)

No Solution

MATLAB Command

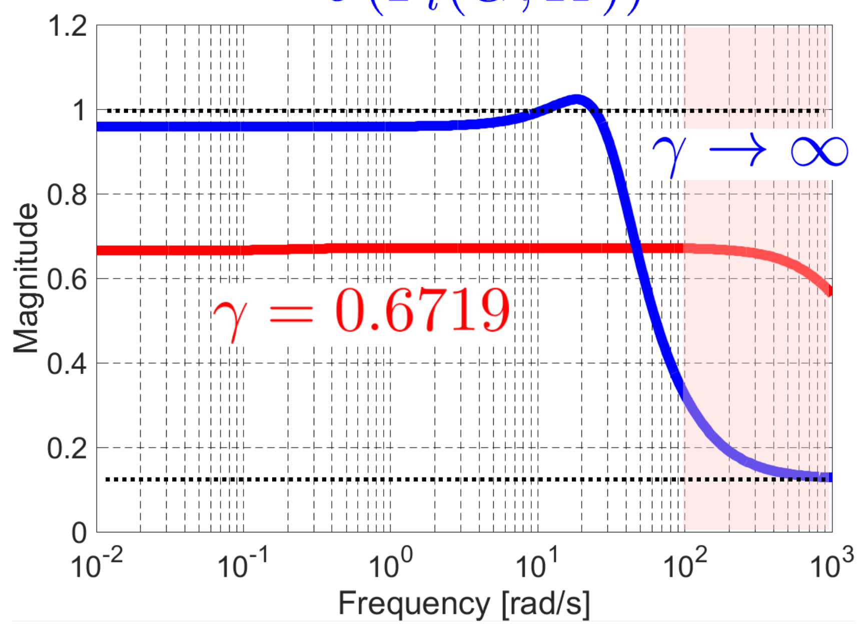
```
[Khi,CLhi,ghi,hiinfo] = …
hinfsyn(G,nmeas,ncon,'Gmax',100,'Gmin',100);
```

MATLAB Command

```
[SV,w]=sigma(CLhi);
figure; semilogx(w,SV)
```

Closed-loop Transfer Function

$\bar{\sigma}(F_l(G,K))$



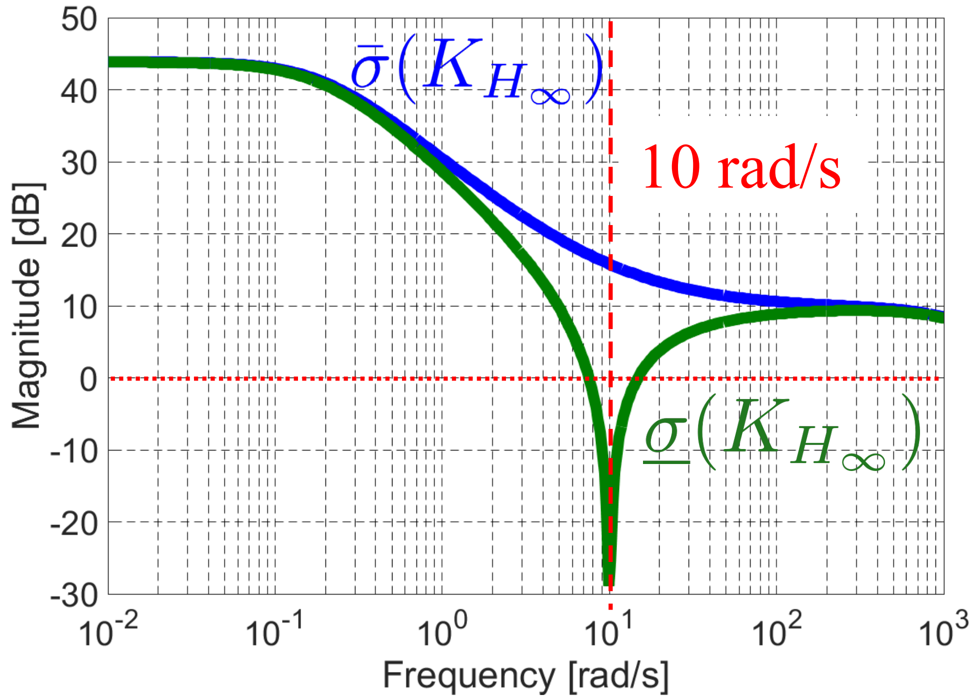
"interested" frequency range

Spinning Satellite: H_∞ Controller

$$K_{H_\infty}(s) = \begin{bmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{bmatrix}$$

MATLAB Command

```
figure
sigma(Khi)
```



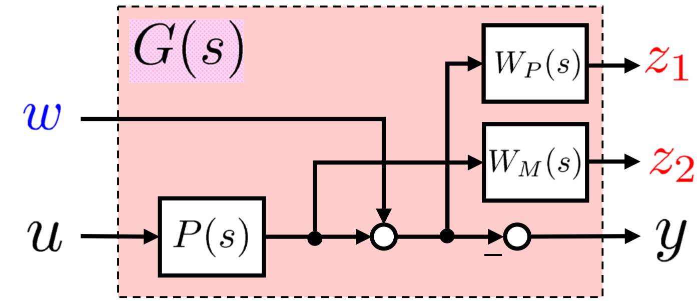
Poles

$-1.0050 \cdot 10^6 \pm 10j,$
$-1.5011 \cdot 10^3 \pm 1.5493 \times 10^{-4}j,$
$-0.1978, -0.1978$

Zeros $-4762, -4762, \pm 10j$

Order 6

Generalized Plant : Order 6



$P(s)$: Order 2

$W_M(s)$: Order 2, $W_P(s)$: Order 2

Nominal Plant Model

Poles $\pm 10j$

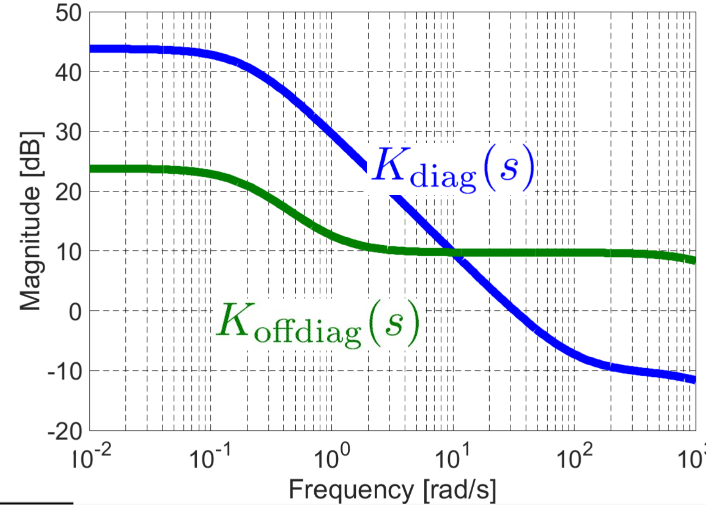
Mixed Sensitivity Problem

⇒ Pole/Zero Cancellations

H_∞ Loop-shaping Design

Spinning Satellite: Beyond SISO Loop Shaping

$$K_{H_\infty}(s) = \begin{bmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{bmatrix}$$



$$K_{\text{diag}}(s) = K_{11}(s) = K_{22}(s) =$$

$$97133 \times \frac{(s+4762)(s-100)}{(s+1501)(s+1978)}$$

$$\times \frac{\cancel{(s+0.1978)}\cancel{(s+1501)}(s+1.005\times10^6)}{\cancel{(s+0.1978)}\cancel{(s+1501)}(s^2+2.01\times10^6 s+1.01\times10^{12})}$$

$$K_{\text{offdiag}}(s) = -K_{12}(s) = K_{21}(s) =$$

$$9.7149\times10^5 \times \frac{(s+4762)(s+1)}{(s+1501)(s+1978)}$$

$$\times \frac{\cancel{(s+0.1978)}\cancel{(s+1501)}(s+1.005\times10^6)}{\cancel{(s+0.1978)}\cancel{(s+1501)}(s^2+2.01\times10^6 s+1.01\times10^{12})}$$

Multiple Phase Lead-lag Controllers

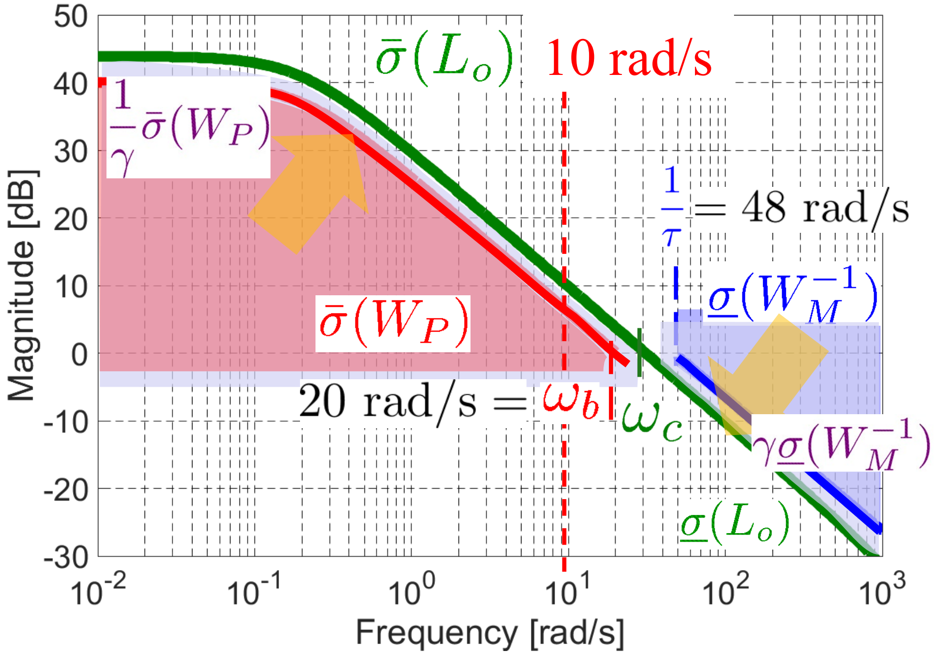
Model Reduction

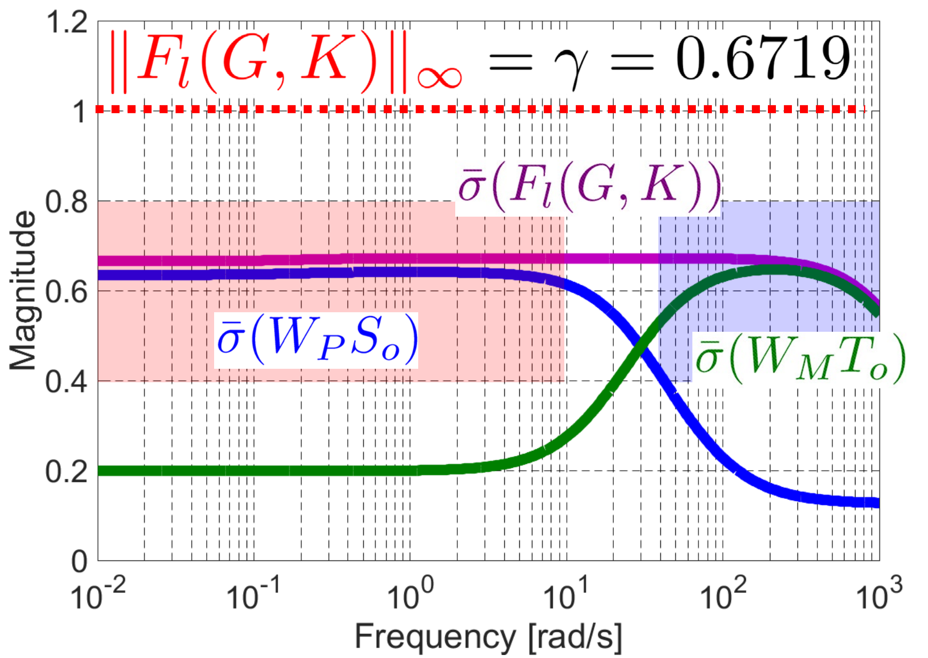
Ex.	[GRED,info] = balancmr(G,order); GRED = hankelmr(G,order);

Discretization

Ex.	sysd = c2d(sysc, Ts, ‘tustin‘); [Ad,Bd,Cd,Dd] = bilin(sys, 1, ‘Tustin‘,Ts);

Spinning Satellite: Open-loop Frequency Response





$\gamma = 0.6719 < 1$

(corresponding maximum stability margin)

Loop Transfer Function

MATLAB Command

```
figure
sigma(Fhi.Lo,WP, inv(WM),WP/ghi,ghi*inv(WM))
```

NP/RS Test

Nominal Performance (NP)

$\|W_P S_o\|_\infty = 0.6411 < 1$

Robust Stability (RS)

$\|W_M T_o\|_\infty = 0.6468 < 1$

MATLAB Command

```
[SV,w]=sigma(WP*Fhi.So);
figure; semilogx(w,SV)
[SV,w]=sigma(WM*Fhi.To);
figure; semilogx(w,SV)
```

Spinning Satellite: Closed-loop Performance

Nominal Stability (NS)

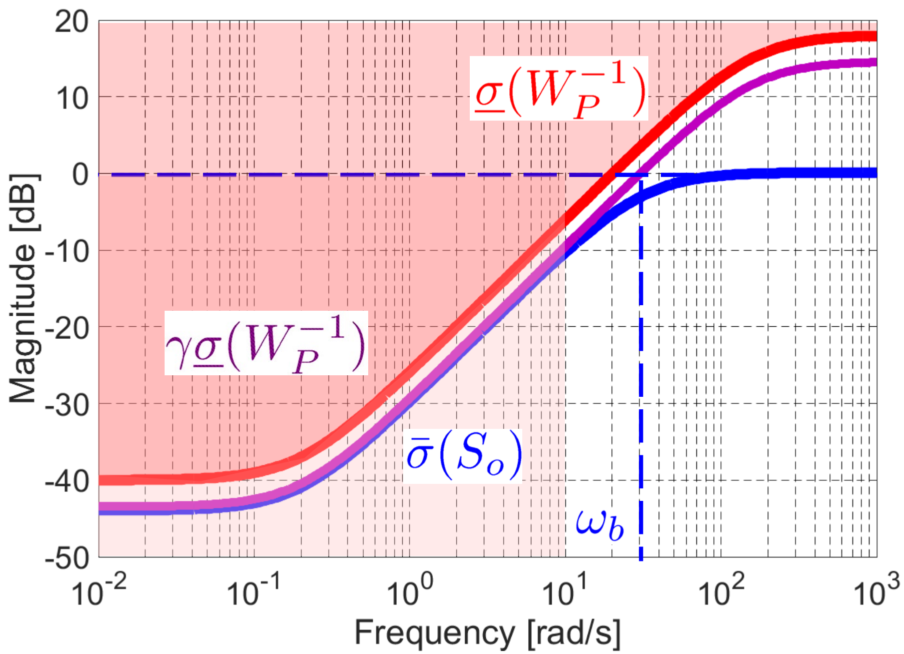
Poles of $F_l(G,K)$

$$p = -0.2, -0.2, -4762, -4762, \\ -1.005 \cdot 10^6 \pm 10j, \pm 10j, \\ -1.480 \cdot 10^3 \pm 0.0j, -31.63 \pm 0.0j$$

MATLAB Command
pole(CLhi) zero(CLhi) figure; pzmap(CLhi)

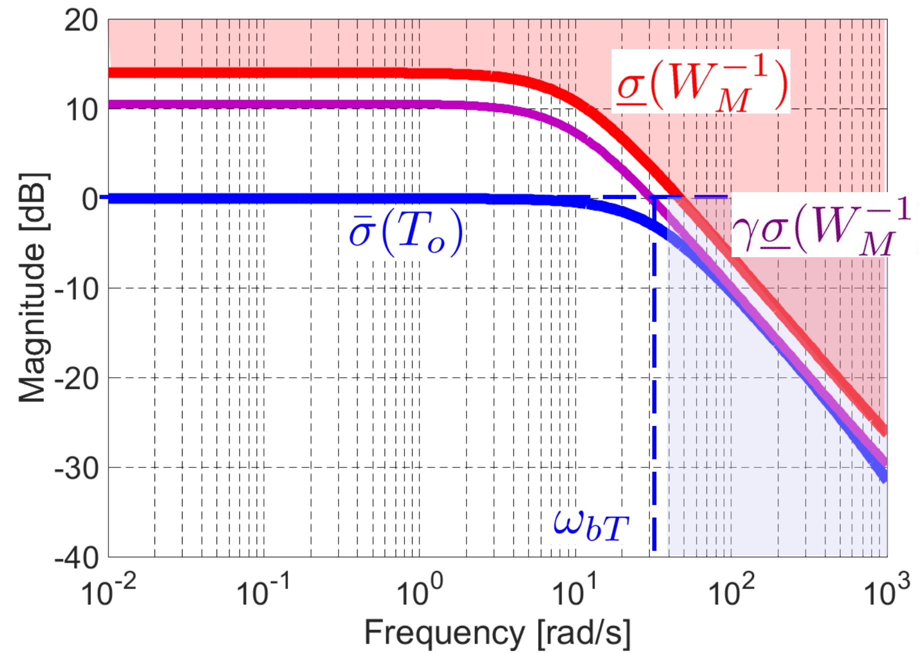
$\gamma = 0.6719 < 1$

Nominal Performance (NP)



MATLAB Command
figure; sigma(Fhi.So,inv(WP),ghi*inv(WP))

Robust Stability (RS)

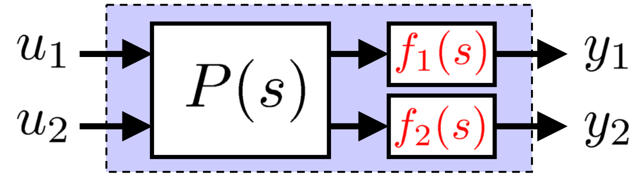


MATLAB Command
figure; sigma(Fhi.To,inv(WM),ghi*inv(WM))

Spinning Satellite: Time Responses for Closed-loop Systems

Perturbation Models

$$\tilde{P}(s) = E_o(s)P(s) = \begin{bmatrix} f_1(s) & 0 \\ 0 & f_2(s) \end{bmatrix} P(s)$$



$$f_i(s) = k_i \frac{-\frac{\theta_i}{2}s + 1}{\frac{\theta_i}{2}s + 1}, \quad i = 1, 2$$

Gain Margin: $0.8 \leq k_i \leq 1.2$

Delay Margin: $0 \leq \theta_i \leq 0.02$

Typical Examples

1) Nominal Model

$k_i = 1, \ \theta_i = 0, \ i = 1, 2$

→ $E_1 = I$

2) Uncertain Gain

$$E_2 = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.8 \end{bmatrix}$$

3) Time Delay

$$E_3 = \begin{bmatrix} f_a & 0 \\ 0 & f_a \end{bmatrix}, \quad f_a = \frac{-0.01s + 1}{0.01s + 1}$$

4) Mixed Perturbation $E_4 = E_2 E_3$

Manual Selection

MATLAB Command

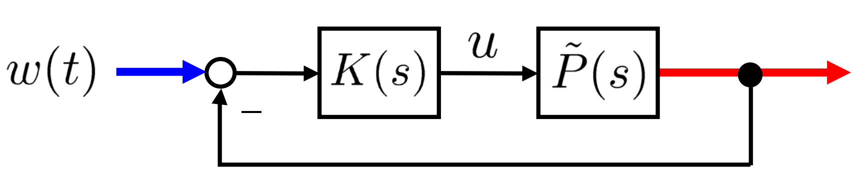
```
k1 = 1.2 ;
k2 = 0.8 ;
L1 = 0.01 ;
L2 = 0.02 ;
f1 = k1*tf([-L1/2 1],[L1/2 1]);
f2 = k2*tf([-L2/2 1],[L2/2 1]);
Eo = diag([f1 f2]);

Pr = Eo*Pnom ;
F = loopsens(Pr,Khi);
```

Automatic Selection

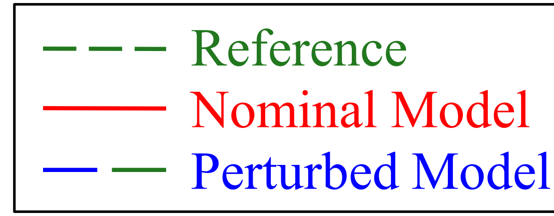
MATLAB Command

```
k1 = ureal('k1',1,'Per',[-20 20]);
k2 = ureal('k2',1,'Per',[-20 20]);
L1 = ureal('L1',0.01,'Range',[0 0.02]);
L2 = ureal('L2',0.01,'Range',[0 0.02]);
f1 = k1*tf([-L1/2 1],[L1/2 1]);
f2 = k2*tf([-L2/2 1],[L2/2 1]);
E = [f1 0;0 f2];
Earray = usample(E,100);
Parray = Earray*Pnom;
Farray = loopsens(Parray,Khi) ;
```

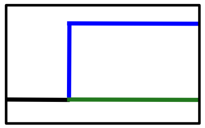
Spinning Satellite: Time Responses for Closed-loop Systems



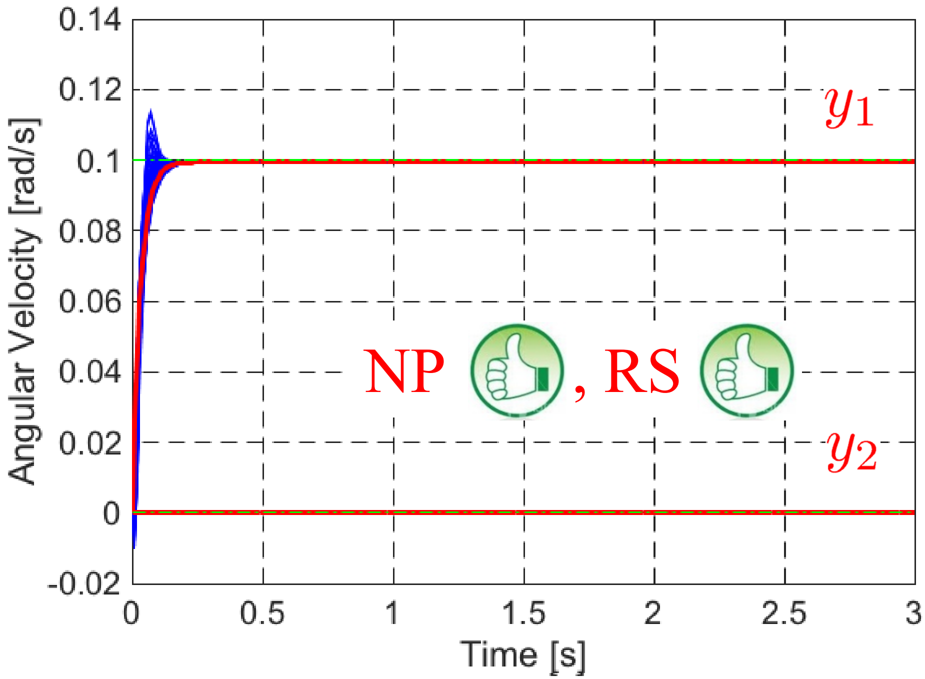
$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

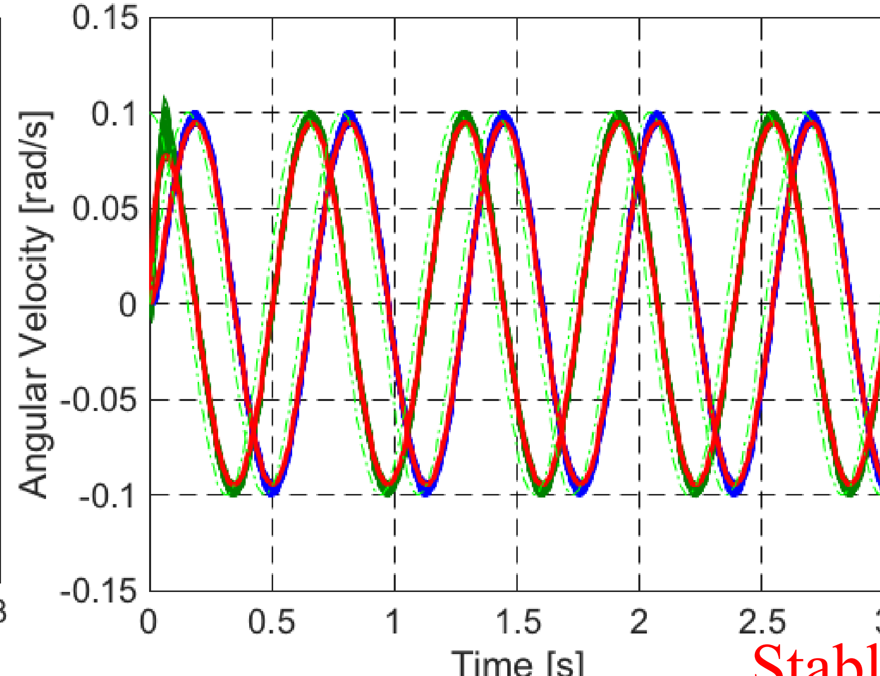


$$w(t) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$



$$w(t) = \begin{bmatrix} 0.1\sin(\omega t) \\ 0.1\cos(\omega t) \end{bmatrix} \quad \omega = 10 \text{ rad/s}$$





Stable

Manual Selection

```
[yhi,t] = lsim(F.To,ref,time);
plot(t,yhi(:,1),'b-'); plot(t,yhi(:,2),'g-');
```

Automatic Selection

```
for i = 1 : 100
   [yhi,t] = lsim(Farray.To(:,:,i),ref,time);
   plot(t,yhi(:,1),'b-'); plot(t,yhi(:,2),'g-');
end
```

6. Design Example 1

- 6.1 Spinning Satellite: H_∞ Control [SP05, Sec. 3.7]
- 6.2 2nd Report

Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.

7. Robust Performance

7.1 Robust Performance [SP05, Sec. 7.6, 8.3, 8.4, 8.10]

7.2 Structured Singular Value μ

[SP05, Sec. 8.5, 8.6, 8.8, 8.11]

7.3 μ-Analysis and Synthesis

[SP05, Sec. 7.6, 8.9, 8.10, 8.11]

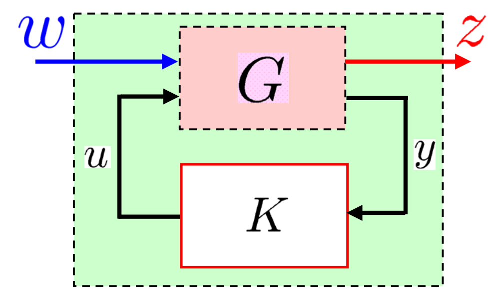
Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.

Improving Performance: Ensuring Assumptions

Generalized Plant

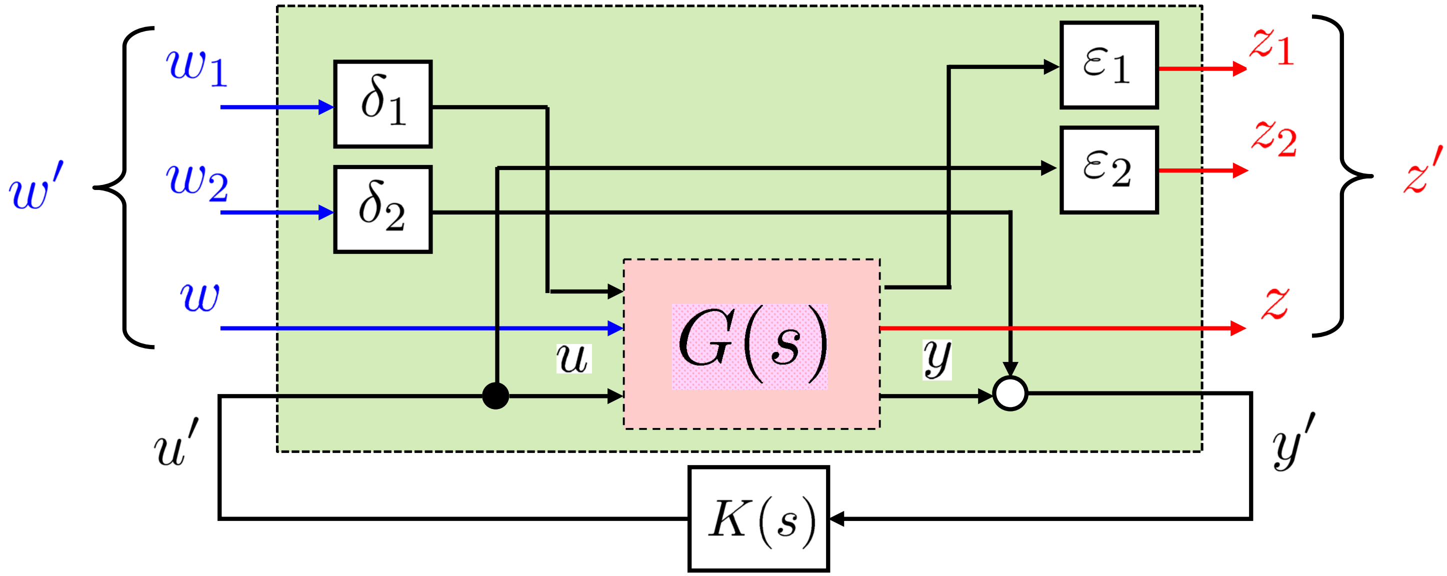
$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$



(A1) (A, B_2) is stabilizable and (C_2, A) is detectable

(A2) (A, B_1) is controllable and (C_1, A) is observable

[Ex.]



Spinning Satellite: Zero Steady-state Tracking Error

$$W_P(s) = w_p(s)I_2\,,\ w_p(s) = \frac{\frac{1}{M_S}s+\omega_b}{s+\omega_b A} \to \frac{\frac{1}{M_S}s+\omega_b}{s}$$

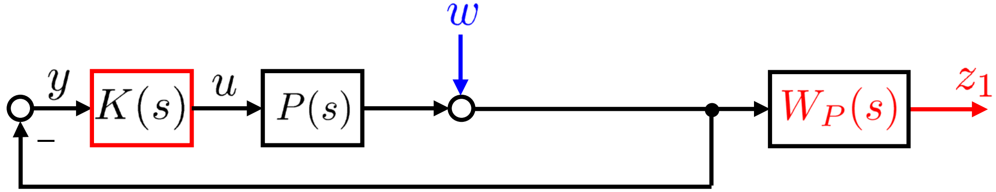
Specifications (Servo System)

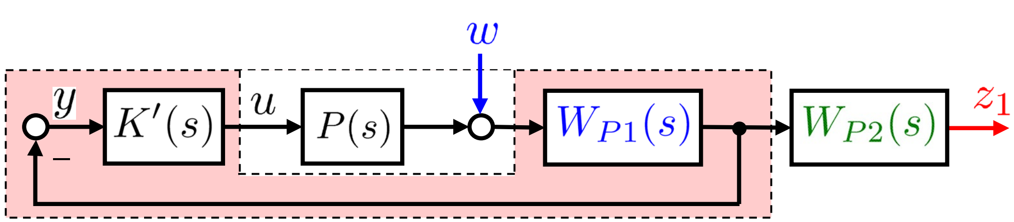
- Let the tracking error for a reference be zero at steady-state → $A = 0$

Assumption (A1) (A, B_2) is stabilizable and (C_2, A) is detectable

NOT satisfied

[Ex.] Modified H_∞ Control Problem





$$W_P(s) = W_{P2}(s)W_{P1}(s)$$

$$K(s) = K'(s)W_{P1}(s)$$

$$W_{P1}(s) = w_{p1}(s)I_2,$$

$$w_{p1}(s) = \frac{s+a}{s} \qquad a > 0$$

$$W_{P2}(s) = w_{p2}(s)I_2,$$

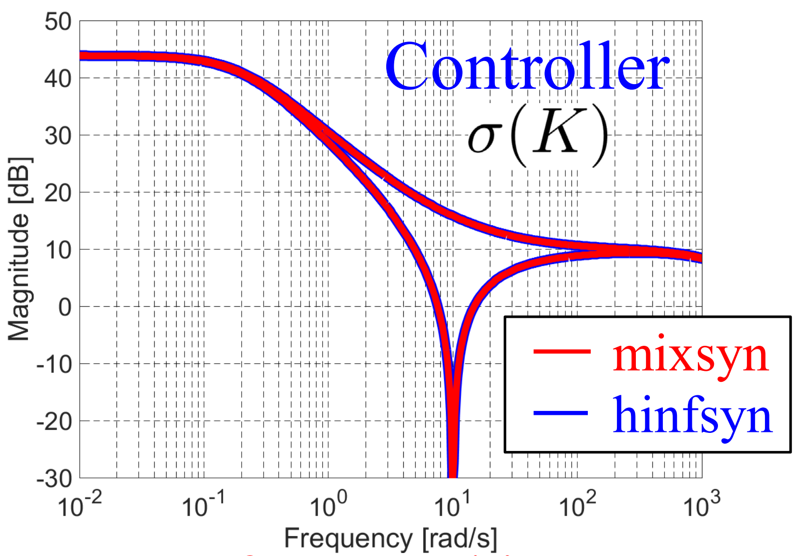
$$w_{p2}(s) = \frac{\frac{1}{M_S}s+\omega_b}{s+a}$$

H_∞ Controller mixsyn (Spinning Satellite)

```
[K, CL, gam, info] = mixsyn( P, WP, WU, WM, key1, value1, …)
```

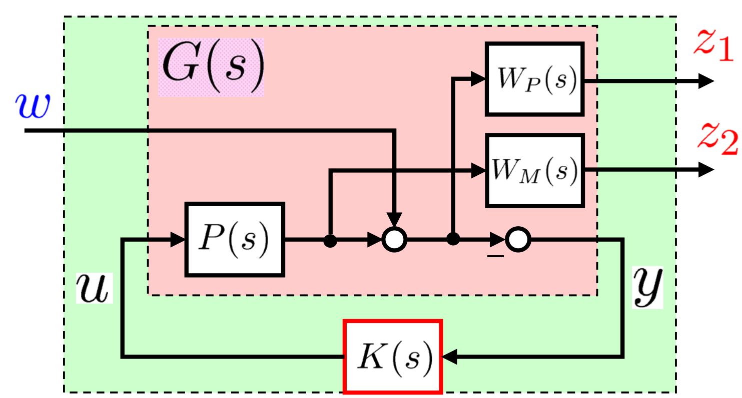
Remark

1. $P(s)$, $W_P(s)$, $W_u(s)$, $W_M(s)$: proper, $W_P(s)$, $W_u(s)$, $W_M(s)$: stable
   $P(s)$: stabilizable and detectable
2. Each of $W_P(s)$, $W_u(s)$ and $W_M(s)$ must be either
   a) empty (you may simply assign an empty matrix "[]"),
   b) scalar (SISO) or
   c) have respective input dimensions n_y, n_u and n_y where P is n_y-by-n_u



Perfect matching

MATLAB Command
[Km,CLm,gm,minfo] = mixsyn(Pnom,WP,[],WM);



Spinning Satellite: High Frequency Roll-off

Gamma Tolerance $\epsilon_\gamma \geq |\gamma - \gamma_{opt}|$

$\epsilon_\gamma = 0.1 \quad \gamma = 0.6875$

$\epsilon_\gamma = 0.01 \quad \gamma = 0.6719$

$\epsilon_\gamma = 0.001 \quad \gamma = 0.6655$

$\epsilon_\gamma = 0.0001 \quad \gamma = 0.6650$

$\epsilon_\gamma = 0.00001 \quad \gamma = 0.6650$

All-pass Property:

necessity to consider
frequency ranges to be controlled

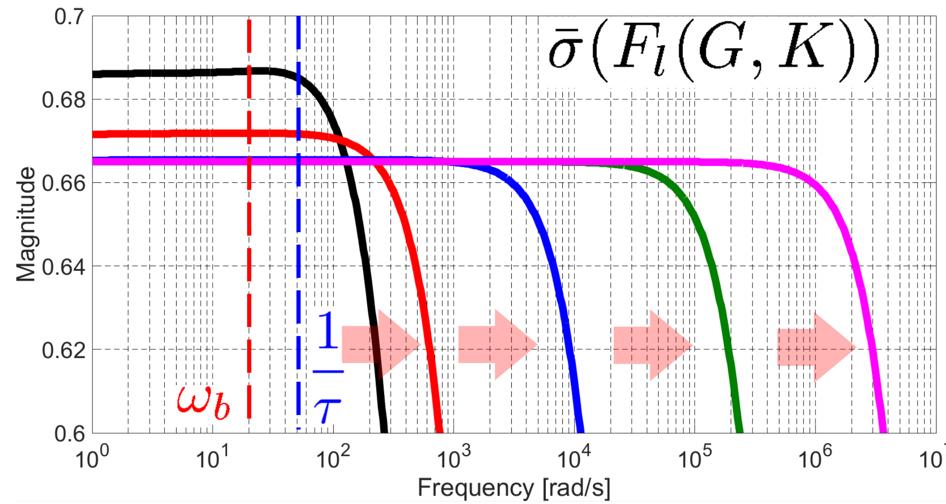
Gamma Tolerance $\epsilon_\gamma \geq |\gamma - \gamma_{opt}|$

$\text{tolgam} = 0.01 \rightarrow 0.1$

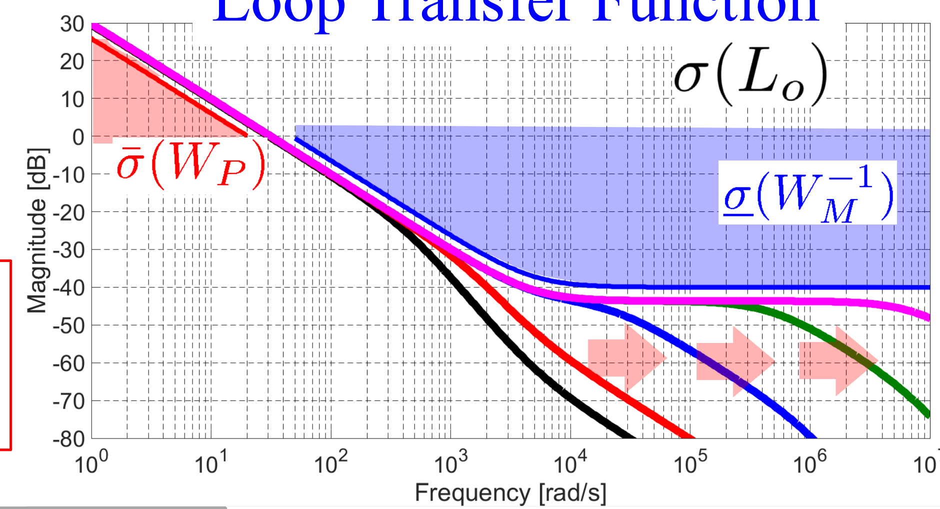
Appropriately sub-optimal

$\gamma = 0.6875 < 1 \; (\gamma_{opt} = 0.6650)$

Closed-loop Transfer Function



Loop Transfer Function



MATLAB Command

```
[Khi,CLhi,ghi,hiinfo] = hinfsyn(G,nmeas,ncon,'Tolgam',0.1);
```

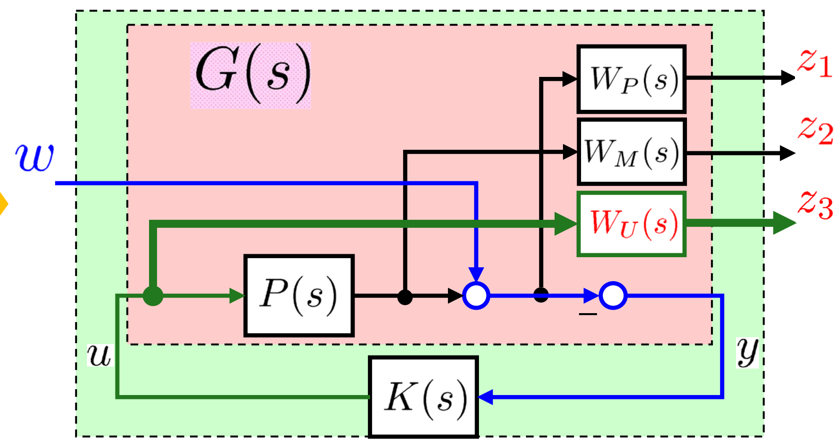
H_2 Controller h2syn (Spinning Satellite)

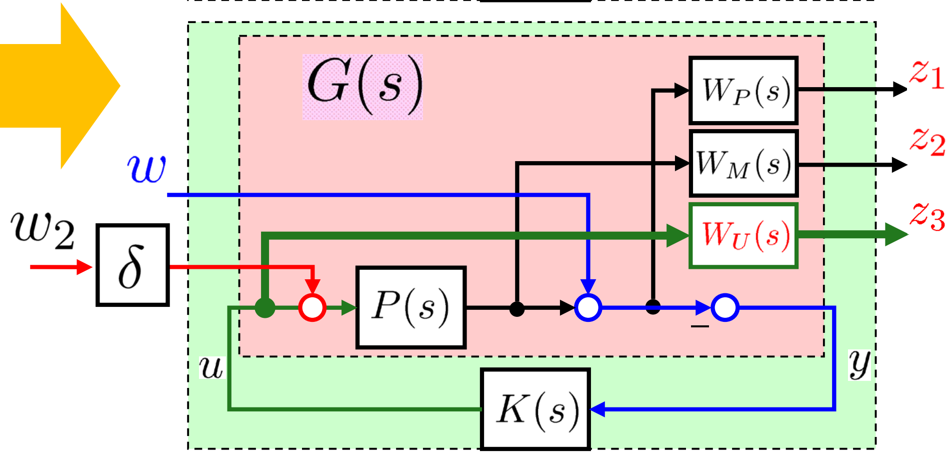
[K, CL, gam, info] = h2syn(G, nmeas, ncon)

Remark (1) (A, B_2) :stabilizable, (C_2, A) :detectable

(2) D_{12} : full column rank, D_{21} : full row rank

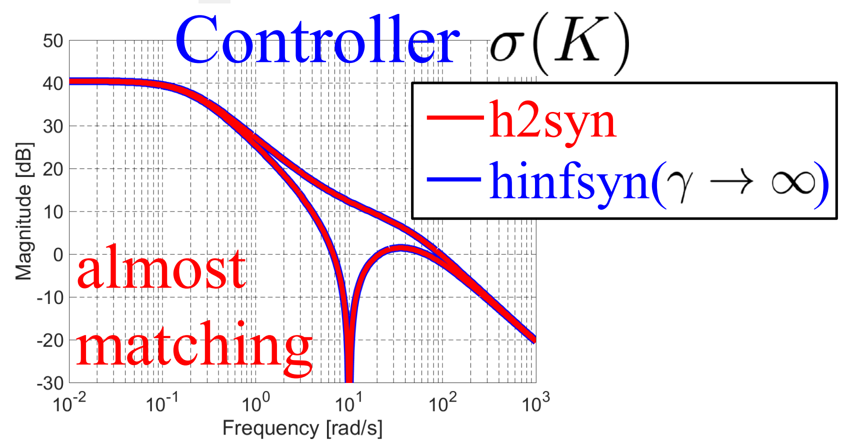
$u \rightarrow z$ $\quad w \rightarrow y$





Error

```
コマンド ウィンドウ
エラー  -
行列の次元は一致しなければなりません。

エラー lti/h2syn (line 194)
    h2 = -b1*d21' - h2c'; % h2 = -h2aprime'

エラー Untitled2 (line 153)
[Khi,CLhi,ghi,hiinfo] = h2syn(G,nmeas,ncon);
```

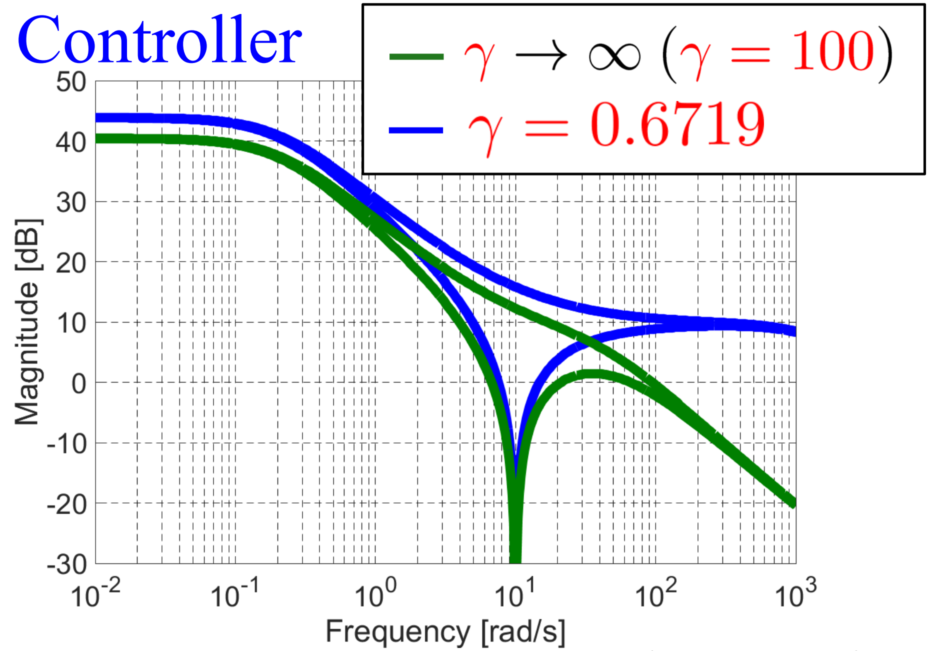


OK $\|F_l(G, K)\|_2 = \gamma = \infty$

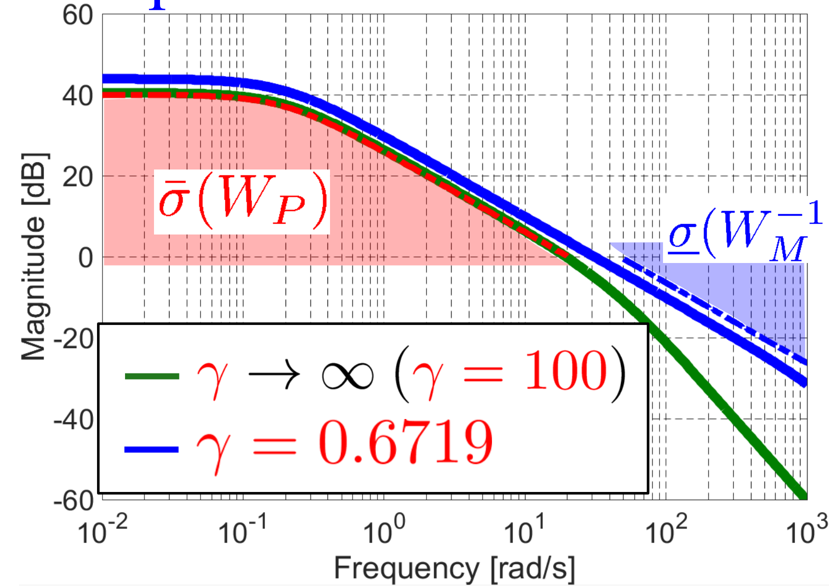
Set $\delta = W_u = 0.0001 I_2$

Spinning Satellite: γ-iteration; $\gamma \to \infty$ vs. $\gamma = 0.6719$

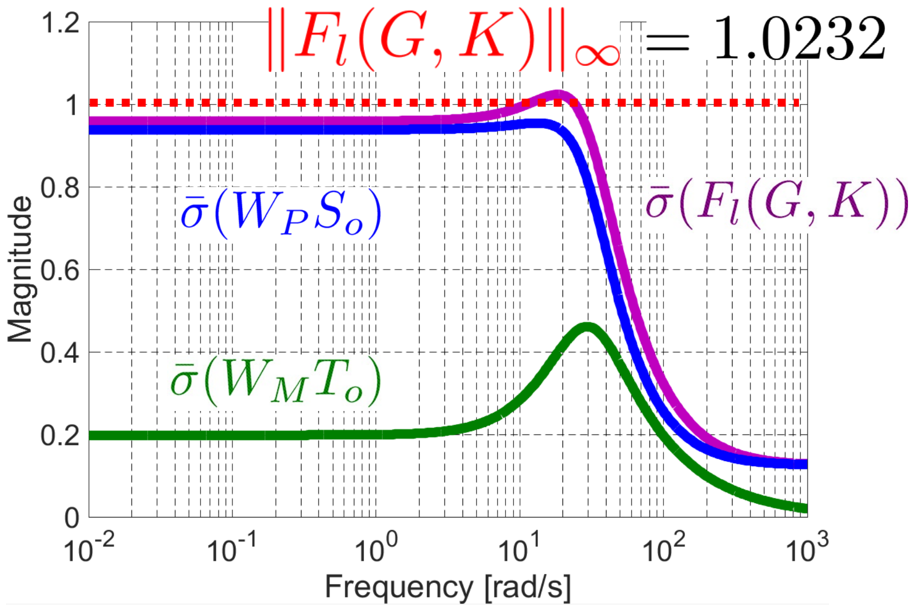
Controller



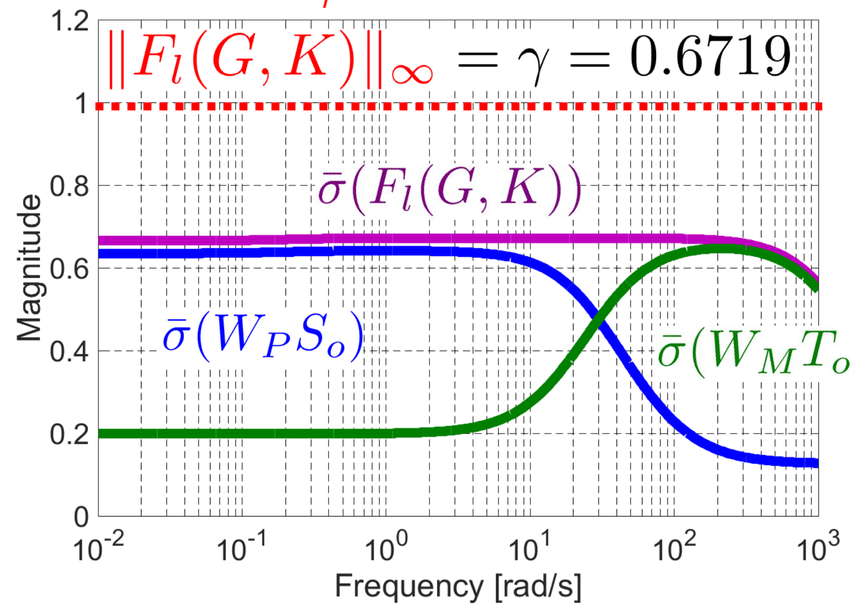
Loop Transfer Function



Closed-loop TF $\gamma \to \infty$ ($\gamma = 100$)

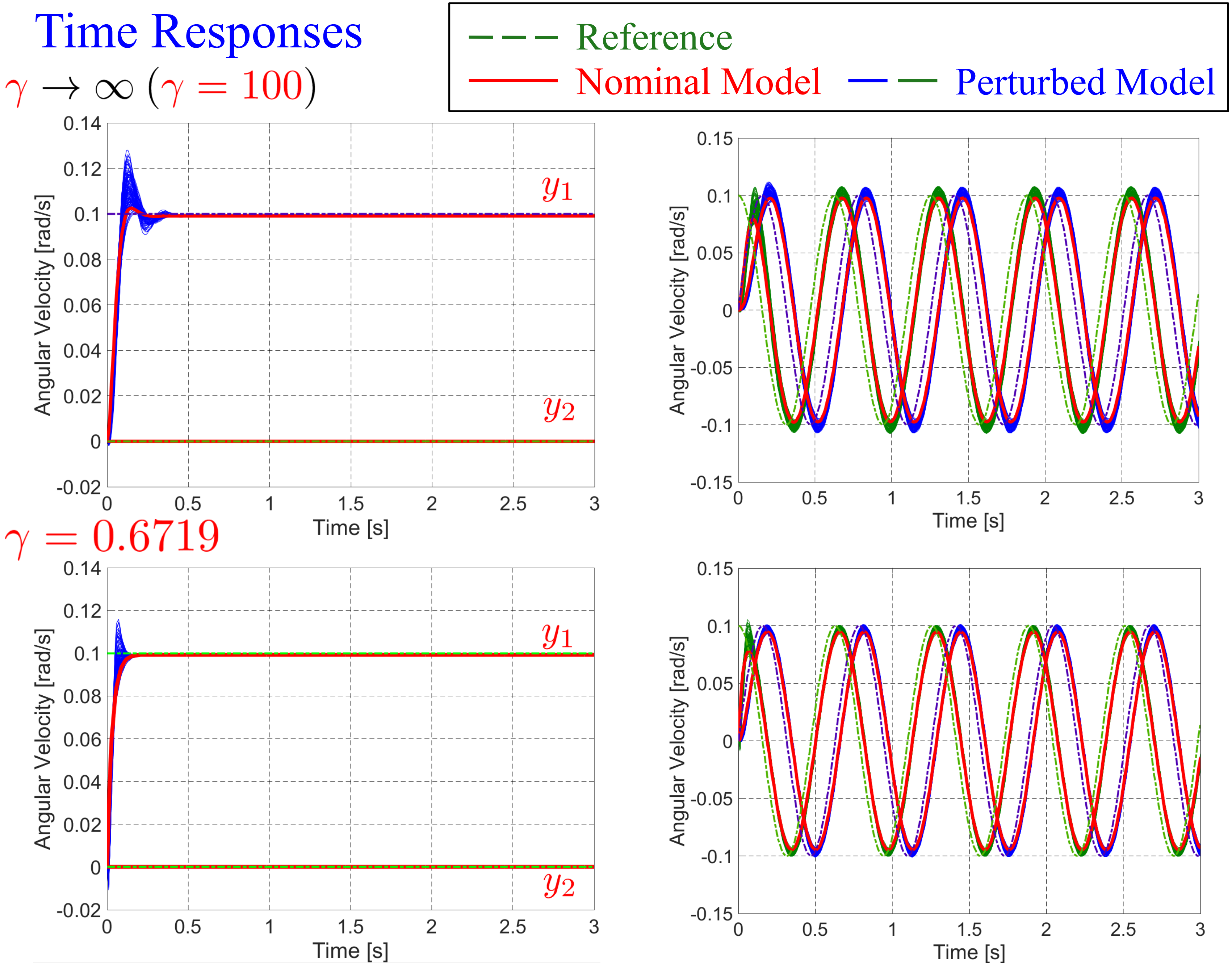


$\gamma = 0.6719$

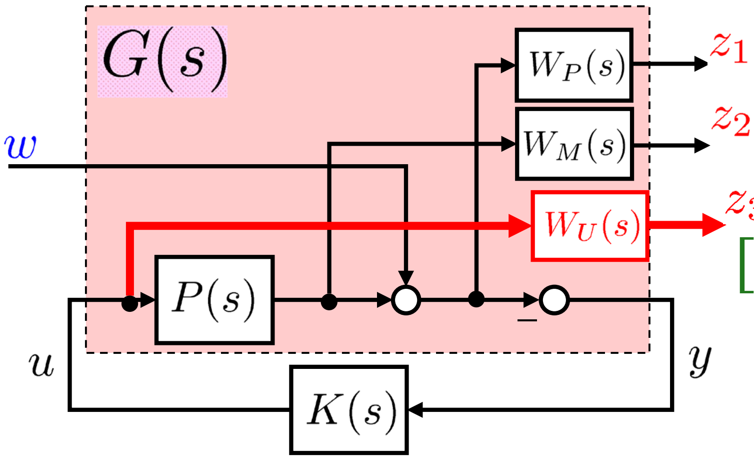


Spinning Satellite: γ-iteration; $\gamma \to \infty$ vs. $\gamma = 0.6719$

Time Responses



Spinning Satellite: Input Weight



$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} W_P(s)S_o(s) \\ -W_M(s)T_o(s) \\ -W_U(s)K(s)S_o(s) \end{bmatrix} w$$

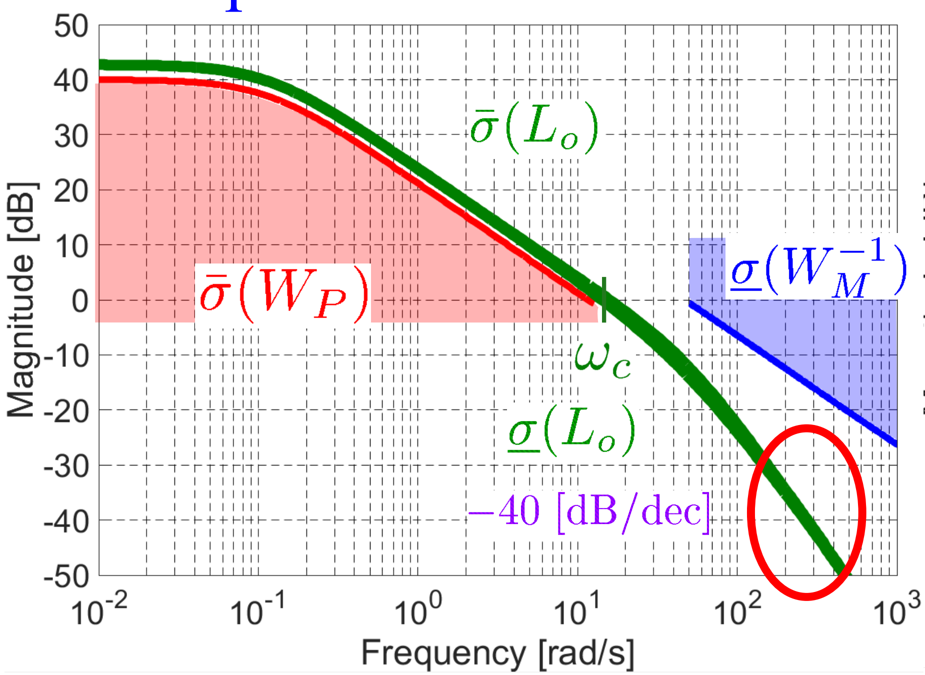
[Ex.] $W_P(s) = w_p(s)I_2\,,\; w_p(s) = \dfrac{0.125s + 11.5}{s + 0.115}$

$\left(\omega_b = 11.5\,,\; M_s = 8,\; A = 0.01\right)$

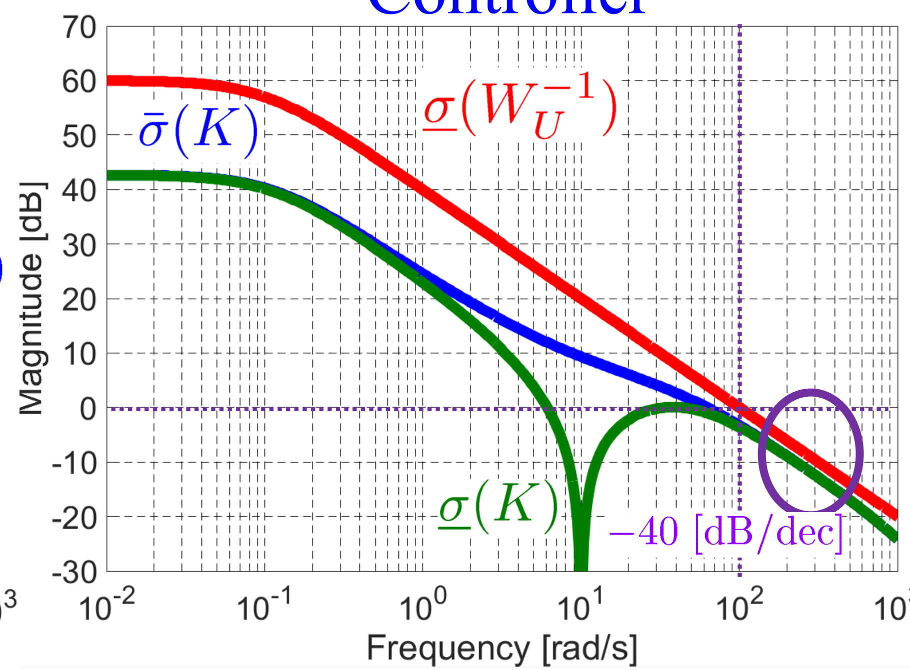
$W_U(s) = w_u(s)I_2\,,\; w_u(s) = \dfrac{10s + 1}{0.05s + 1000}$

→ $\gamma = 0.7578$ NP ○, RS ○

Loop Transfer Function



Controller



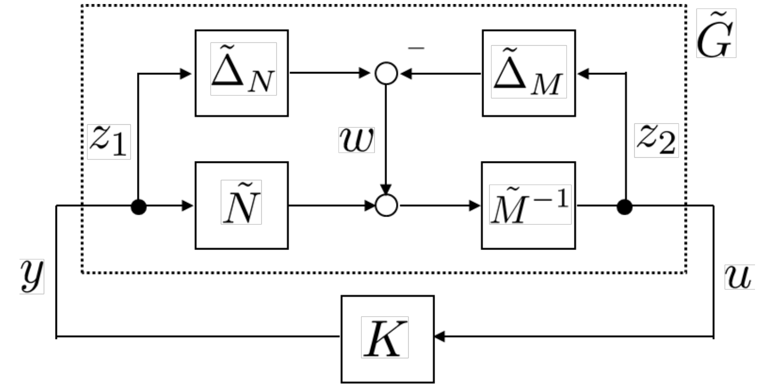
H_∞ Loop-Shaping Design [SP05, Sec. 9.4]

Coprime Factor Uncertainty

$$G = M^{-1}N \quad \tilde{G} = \left(\tilde{M} + \tilde{\Delta}_M\right)^{-1}\left(\tilde{N} + \tilde{\Delta}_N\right)$$

LFT closed loop system

$$F_l(G,K) = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1}\tilde{M}^{-1} \right\|_\infty$$
$$= \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} I & G \end{bmatrix} \right\|_\infty$$



Robust Stability Condition

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} I & G \end{bmatrix} \right\|_\infty < \frac{1}{\epsilon} = \gamma$$

H_∞ Loop-shaping design procedure (LSDP)

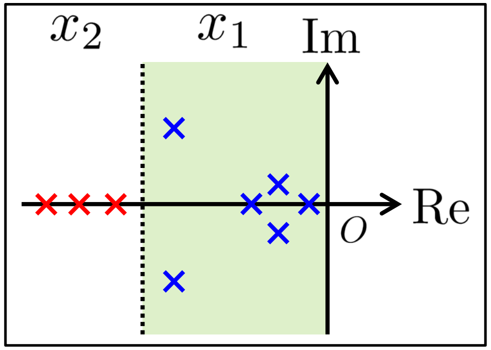
loopsyn   ncfsyn

D. McFarlane and K. Glover, Springer-Verlag, 1990

(See Advanced Document in details)

Model Reduction [SP05, Sec. 11]

Truncation and Residualization [SP05, Sec. 11.2]



Jordan form

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \Rightarrow \begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u \\ y = C_1 x_1 + C_2 x_2 + Du \end{cases}$$

G : stable $x \in \mathbb{R}^n$ $\quad x^T = [\, x_1^T \; x_2^T \,]^T, \; x_1 \in \mathbb{R}^k$

$$A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)\,, \quad |\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|\,,$$

$B = [b_i^T]\,,\ C = [c_i]$ ⇒ x_2 contains the fast modes

MATLAB Command
[Gs,Gf] = slowfast(G,ns);

Modal Truncation $(x_2 \to 0)$

MATLAB Command
[SysG1,SysG2] = modreal(G,cut);

$$G_a = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] \quad G(j\infty) = G_a(j\infty) = D$$

$$G - G_a = \sum_{i=k+1}^{n} \frac{c_i b_i^T}{s - \lambda_i} \Rightarrow \|G - G_a\|_\infty \le \sum_{i=k+1}^{n} \frac{\bar{\sigma}(c_i b_i^T)}{|\mathrm{Re}(\lambda_i)|}$$

Model Error

Residualization $(\dot{x}_2 \to 0)$

$$G_a = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] \quad \begin{cases} A_r = A_{11} - A_{12}A_{22}^{-1}A_{21} \\ B_r = B_1 - A_{12}A_{22}^{-1}B_2 \\ C_r = C_1 - C_2 A_{22}^{-1}A_{21} \\ D_r = D - C_2 A_{22}^{-1}B_2 \end{cases}$$

- $G(0) = G_a(0)$
- Equivalent to Singular Perturbational Approximation

Model Reduction [SP05, Sec. 11]

Balanced Realizations [SP05, Sec. 11.3]

MATLAB Command
[Sysb,g] = balreal(G);

Let $G = (A, B, C, D)$ be a minimal realization of a stable, rational transfer function, then G is called *balanced* if the solutions to the following Lyapunov equations

$$AP + PA^T + BB^T = 0 \quad \text{and} \quad A^TQ + QA + C^TC = 0$$

are $P = Q = \operatorname{diag}(\sigma_1, \cdots, \sigma_n) \triangleq \Sigma$, where $\sigma_1 \geq \cdots \geq \sigma_n > 0$.

Controllability Gramian $\quad P \triangleq \int_0^\infty e^{At}BB^Te^{A^Tt}dt$

Observability Gramian $\quad Q \triangleq \int_0^\infty e^{A^Tt}C^TCe^{At}dt$

$\sigma_i \triangleq \sqrt{\lambda_i(PQ)}$: Hankel singular values cf. [SP05, p. 160]

Σ : Gramian of G , $\quad \sigma_1 = \|G\|_H$: Hankel norm of G

(The size of σ_i is a relative measure of the contribution that x_i makes to the input-output behavior of the system.)

Model Reduction [SP05, Sec. 11]

Balanced Truncation/Residualization [SP05, Sec. 11.4]

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \Rightarrow A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \qquad \begin{array}{l} \Sigma_1 = \mathrm{diag}(\sigma_1, \cdots, \sigma_k) \\ \Sigma_2 = \mathrm{diag}(\sigma_{k+1}, \cdots, \sigma_n) \end{array} \qquad \sigma_k > \sigma_{k+1}$$

MATLAB Command
[GRED,info] = balancmr(G,order);

[SP05, Theorem 11.1] (p. 459)

Let $G(s)$ be a stable rational transfer function with Hankel singular values $\sigma_1 > \cdots > \sigma_n$ where each σ_i has multiplicity r_i and let $G_a^k(s)$ be obtained by truncating or residualizing the balanced realization of $G(s)$ to the first $(r_1 + \cdots + r_k)$ states. Then

$$\|G - G_a^k\|_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i$$

"twice the sum of the tail"

(i) Balanced residualization preserves the steady-state gain of the system

(ii) Balanced residualization is related to balanced truncation by the bilinear transformat $s \to s^{-1}$

Model Reduction [SP05, Sec. 11]

Optimal Hankel Norm Approximation [SP05, Sec. 11.5]

The Hankel norm of any stable transfer function $E(s)$

$$\|E(s)\|_H = \sigma_1 = \sqrt{\rho(PQ)}$$

MATLAB Command

```
GRED = hankelmr(G,order);
```

[SP05, Theorem 11.2] (p. 460)

Let $G(s)$ be a stable, square, transfer function with Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_k \leq \sigma_{k+1} = \cdots = \sigma_{k+l} > \sigma_{k+l+1} \geq \cdots \geq \sigma_n > 0$, then a stable optimal Hankel norm approximation of order k, $G_h^k(s)$, can be constructed as follows.

$$\Sigma = \mathrm{diag}(\Sigma_1, \sigma_{k+1} I)\ , \quad \Sigma_1 = \mathrm{diag}(\sigma_1, \cdots, \sigma_k, \sigma_{k+l+1}, \cdots, \sigma_n)$$

$$G_h^k + F = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \begin{cases} \hat{A} = \Gamma^{-1}(\sigma_{k+1}^2 A_{11}^T + \Sigma_1 A_{11} \Sigma_1 - \sigma_{k+1} C_1^T U B_1^T) \\ \hat{B} = \Gamma^{-1}(\Sigma_1 B_1 + \sigma_{k+1} C_1^T U) \\ \hat{C} = C_1 \Sigma_1 + \sigma_{k+1} U B_1^T \\ \hat{D} = D - \sigma_{k+1} U \end{cases}$$

The Hankel norm of the error between $G(s)$ and $G_h^k(s)$:

$$\|G - G_h^k\|_H = \sigma_{k+1}(G)$$

Model Reduction [SP05, Sec. 11]

Reduction of unstable models [SP05, Sec. 11.6]

1. Stable part model reduction

$$G(s) = G_u(s) + G_s(s)$$
$$G_a(s) = G_u(s) + G_{sa}(s)$$

MATLAB Command
[Gs,Gus,m] = stabproj(G);

(Balanced truncation/residualization or optimal Hankel norm approximation can be applied to the stable part G_s to find a reduced order approximation G_{sa})

2. Coprime factor model reduction

$$G(s) = M^{-1}(s)N(s)$$
$$\Rightarrow G_a(s) = M_a^{-1}(s)N_a(s)$$

MATLAB Command
GRED = ncfmr(G,order);

MATLAB Commands (Model Simplification)

Control System Toolbox

balred , modred,
sminreal, minreal, balreal,

hsvd, hsvplot,
balredOptions, hsvdOptions

Robust Control Toolbox

balancmr, bstmr, dcgainmr,
hankelmr, modreal, ncfmr,
schurmr, slowfast, reduce,

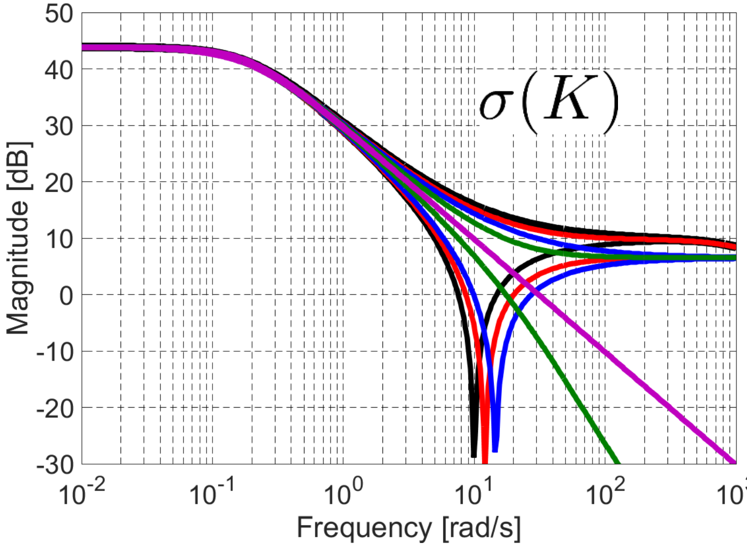
Standard use

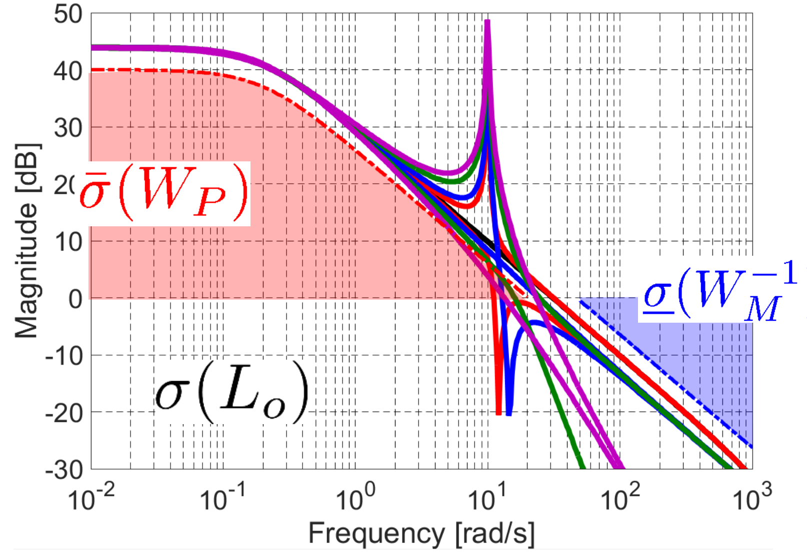
hankelsv

Spinning Satellite: Model Reduction

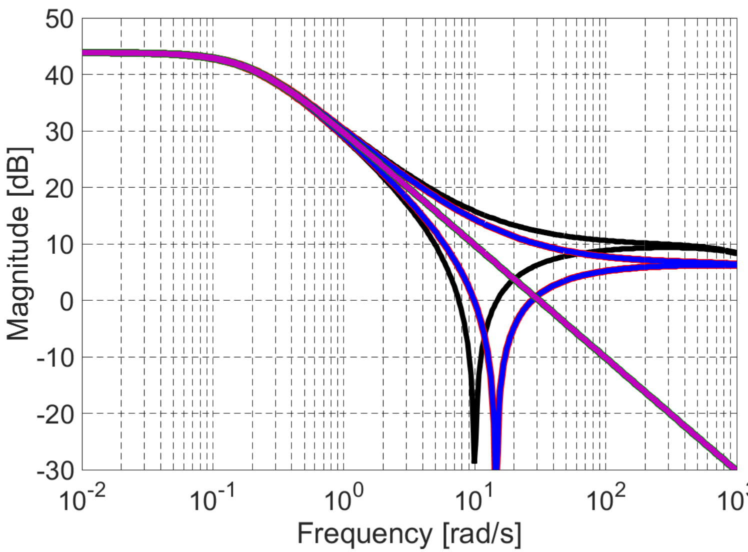
Order 6 (default)
Order 5 Order 4
Order 3 Order 2

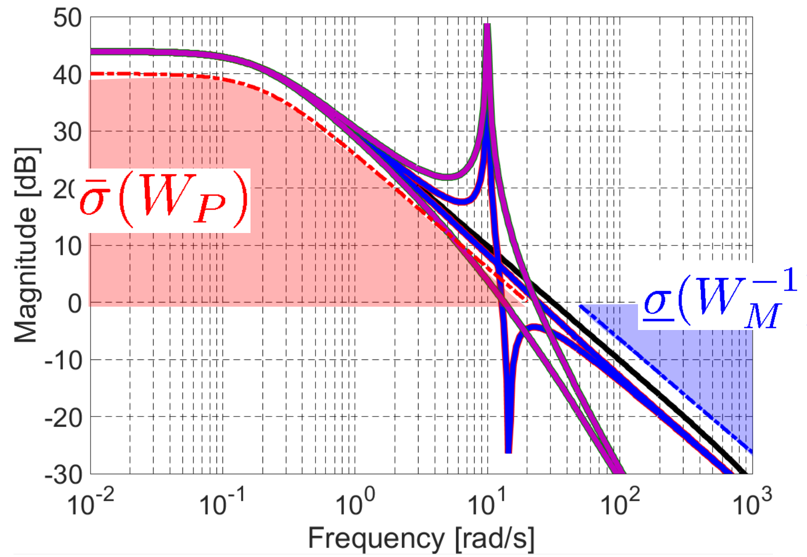
[Ex.] balancmr



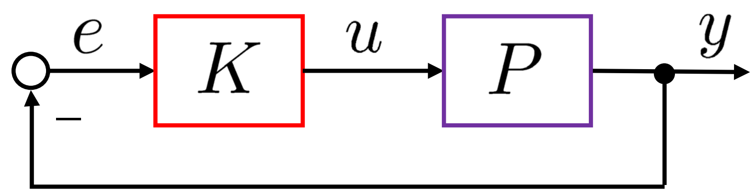


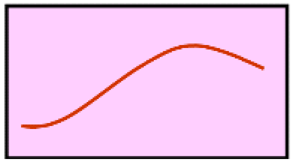
[Ex.] hankelmr

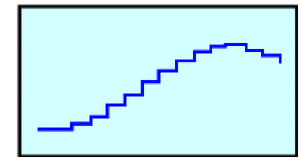




Discretization







Continuous-time Plant $P(s)$ —Discretization→ Discrete-time Plant $P[z]$

Continuous-time Plant $P(s)$ —Design Controller→ Continuous-time Controller $K(s)$

Discrete-time Plant $P[z]$ —Design Controller→ Discrete-time Controller $K[z]$

Continuous-time Controller $K(s)$ —Digital Redesign→ Discrete-time Controller $K[z]$

MATLAB Commands (Continuous-Discrete Conversion)

Continuous to Discrete	Sysd = c2d(Sysc, Ts, method)
Discrete to Continuous	Sysc = d2c(Sysd, method)
Resample Discrete-time model	Sysd2 = d2d(Sysd, Ts, method)

Spinning Satellite: Infeasible Performance Weight

$$\text{Find } K \text{ s.t. } \|F_l(G,K)\|_\infty < \gamma < 1$$

a) $\omega_b = 20 \quad w_p(s) = \dfrac{0.125s + 20}{s + 0.2}$

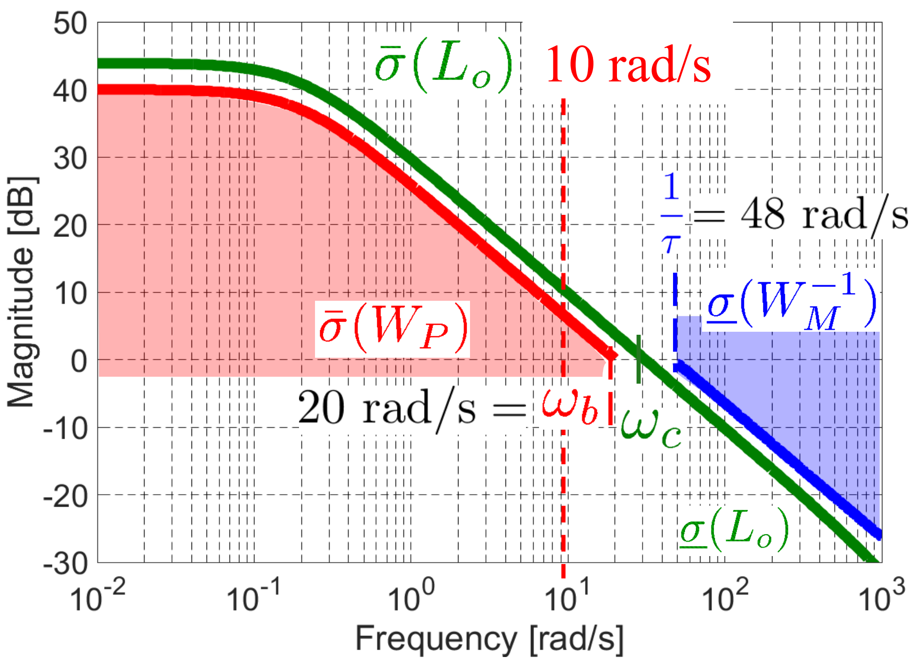
→ $\gamma = 0.6719 < 1$ ○

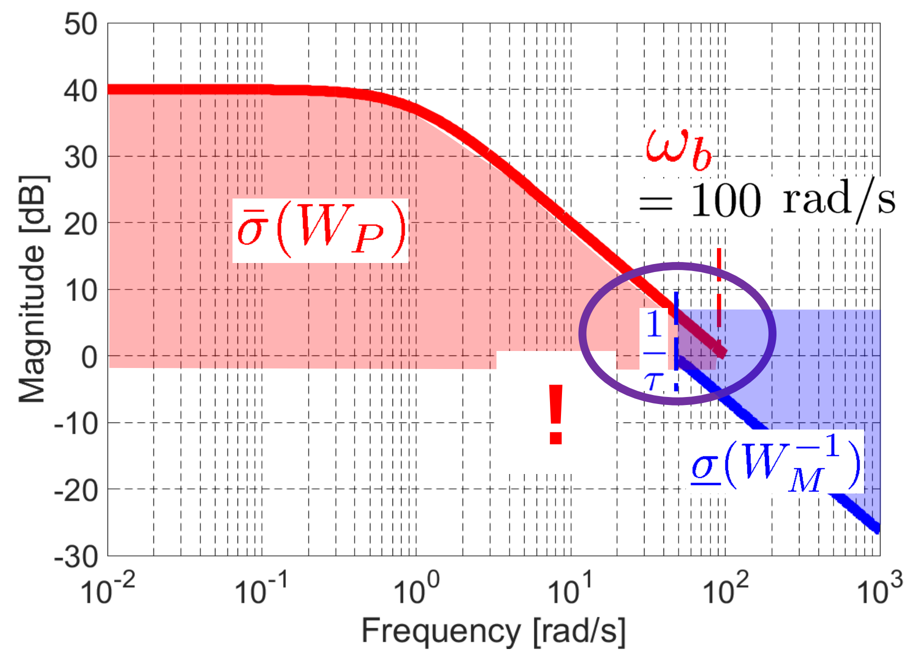
A H_∞ controller exists

b) $\omega_b = 100 \quad w_p(s) = \dfrac{0.125s + 100}{s + 1}$

→ $\gamma = 1.4402 \geq 1$ ×

No controller



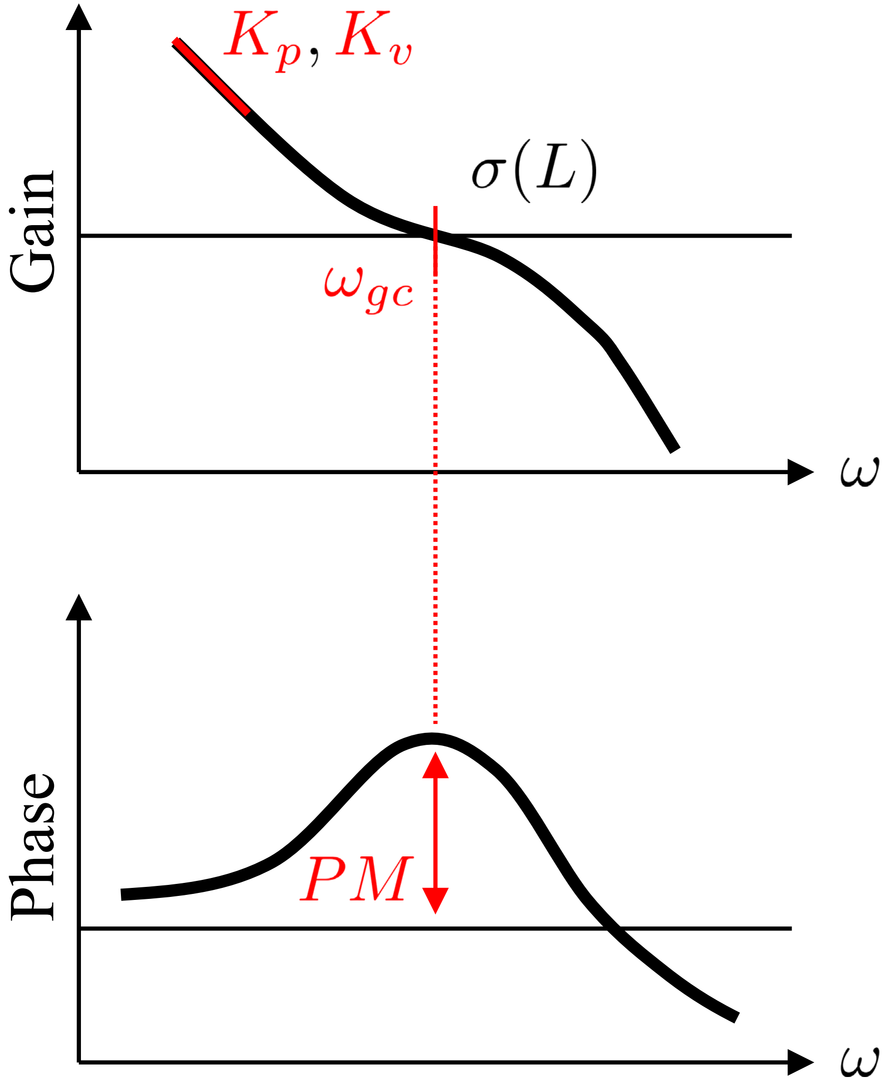


MATLAB Command

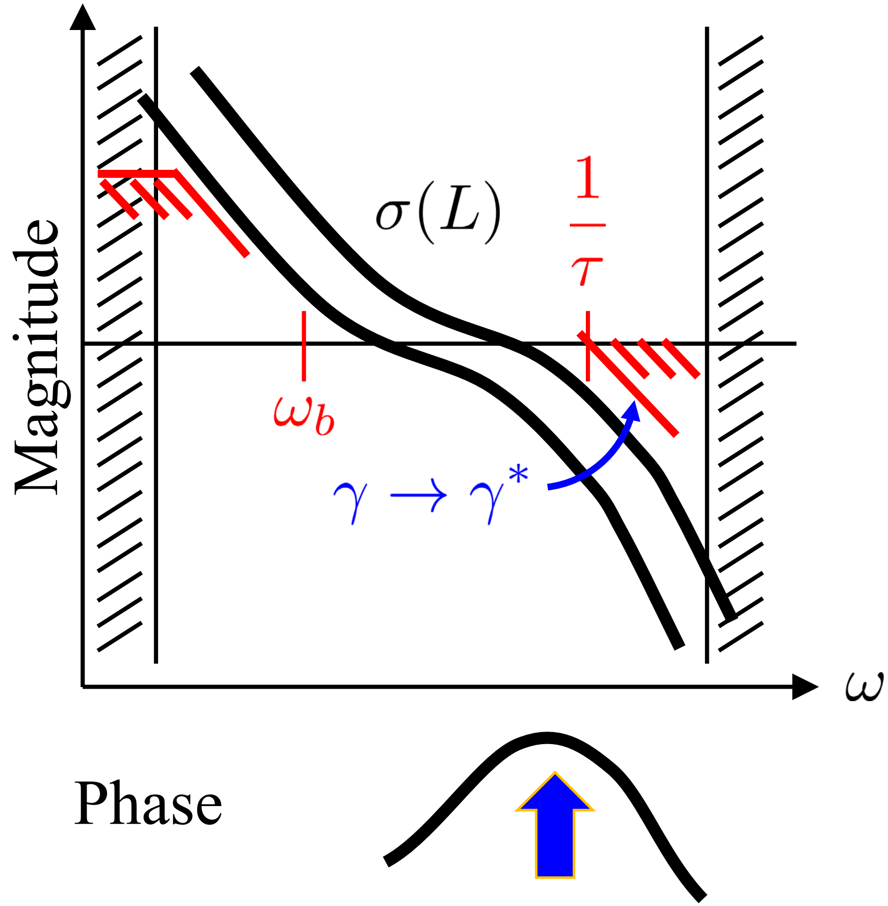
```
[Khi,CLhi,ghi,hiinfo] = hinfsyn(G,nmeas,ncon,'Gmax',1,'Gmin',1);
```

Inclusion of Classical Feedback Control

SISO Loop Shaping



MIMO Loop Shaping



SISO System: Phase-lead Compensator