Robust and Optimal Control

Spring, 2015
Instructor: Prof. Masayuki Fujita (S5-303B)

1st class
Tue., 7th April, 2015, 10:45～12:15,
S511 Lecture Room
1. Nominal Stability and Nominal Performance

1.1 Nominal Stability

1.2 Multivariable Frequency Response Analysis

1.3 Nominal Performance

1.4 Sensitivity Minimization

Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Motivating Example: Spinning Satellite’s Attitude Control

JAXA: ETS-VIII Spinning Satellite

Roll Pitch Yaw

Outputs: $y_1, y_2$ Angular velocity

Inputs: $u_1, u_2$ Torque

2-Input 2-Output System

$\alpha = 10 \text{ rad/s}$
Motivating Example: Spinning Satellite’s Attitude Control

Multivariable Systems

\[ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ -\frac{10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix} \]

Diagonal Plant

\[ P_{11}(s) = P_{22}(s) = \frac{s - 100}{s^2 + 100} \]

Pole: \( s = \pm 10j \) Vibratory System
Zero: \( s = 100 \) Non-minimum Phase System

How to design multivariable feedback controllers systematically?

Interaction (Coupling)
Nominal Stability

Internal Stability of Multivariable Feedback Systems

[SP05, Fig. 4.3] (p. 145)

\[ u = (I + KP)^{-1}d_u - K(I + PK)^{-1}d_y = S_i d_u - KS_o d_y \]
\[ y = P(I + KP)^{-1}d_u - (I + PK)^{-1}d_y = PS_i d_u - S_o d_y \]

[SP05, Theorem 4.6] (p. 145) Nominal Stability (NS) Test

Assume \( P, \ K \) contain no unstable hidden modes.
Then, the feedback system in the figure is **internally stable**
if and only if all four closed-loop transfer matrices are stable
Internal Stability

\[ \begin{align*}
K & \quad d_u \quad u \\
+ & \quad P \\
+ & \quad d_y \\
\rightarrow & \quad y
\end{align*} \]

Closed-loop system

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

[ZD97, Thm. 5.5] The system is internally stable iff \( A \) is stable

Internal Stability in SISO Systems

Gang of Four

Complementary Sensitivity

Sensitivity

\[
S = \frac{1}{1 + PK}
\]

Load Sensitivity

\[
PS = \frac{P}{1 + PK}
\]

Noise Sensitivity

\[
KS = \frac{K}{1 + PK}
\]

Well-posedness: \( 1 + P(\infty)K(\infty) \neq 0 \) (Gang of Four: well-defined and proper)

All Stabilizing Controllers (Stable)

Case 1: Stable MIMO Plant $P(s)$ [SP05, p. 148]

All Stabilizing Controllers:

$$K(s) = (I - QP)^{-1}Q = Q(I - PQ)^{-1}$$

$Q$ - (Youla) Parameterization

$Q$ : Stable transfer function matrix

Ex. Gang of Four in SISO Systems (Affine in $Q$)

$$S = \frac{1}{1 + PK} = 1 - PQ$$

$$PS = \frac{P}{1 + PK} = P(1 - PQ)$$

$$T = \frac{PK}{1 + PK} = PQ$$

$$KS = \frac{K}{1 + PK} = Q$$
All Stabilizing Controllers (Unstable)*

Case 2: Unstable MIMO Plant $P(s)$ [SP05, p. 149]

Left Coprime Factorization [SP05, p. 122] (can be also on the right)

$$P(s) = M_l^{-1} N_l \quad M_l, N_l : \text{Stable coprime trans. func. matrices}$$

iff Bezout Identity $N_l X_r + M_l Y_r = I \quad X_r, Y_r : \text{Stable matrices}$

All Stabilizing Controllers

$$K(s) = (Y_r - Q N_l)^{-1} (X_r + Q M_l)$$

$Q$: Stable trans. func. matrix satisfying $\det(Y_r(\infty) - Q(\infty) N_l(\infty)) \neq 0$

State-Space Computation

(State Feedback + Observer + $Q$)
Multivariable Frequency Response Analysis

Frequency Response for SISO Systems

[Ex.]

\[ g(s) = \frac{1}{s^2 + 0.5s + 1} \quad \omega_n = 1 \quad \zeta = 0.25 \]

\[ u(t) = \sin(\omega t) \quad y(t) = |g(j\omega)|\sin(\omega t + \angle g(j\omega)) \]

\[ \max_{\omega} |g(j\omega)| = 2.066 \]

\[ \max_{\omega} |g(j\omega)| = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} = 2.066 \]
Frequency Response for MIMO Systems

[Ex.] \[ G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix} \]

\[ u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \]

\[ \omega = 0.1 \]

\[ \omega = 1 \]

SISO \[ \max_\omega |g(j\omega)| \rightarrow \text{ MIMO} \quad ? \]
Singular Value Decomposition [SP05, A.3]

[SP05, Ex. 3.3] (p. 74) \( \text{svd}(G) \)

\[
G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \quad \Rightarrow \quad G = \begin{bmatrix} -0.87 & -0.48 \\ -0.48 & 0.87 \end{bmatrix} \begin{bmatrix} 7.34 & 0 \\ 0 & 0.27 \end{bmatrix} \begin{bmatrix} -0.79 & -0.60 \\ -0.60 & -0.79 \end{bmatrix}
\]

\( \|u\|_2 = 1 \)

\[
\tilde{\sigma}(G) = \max_{\|u\|_2=1} \|Gu\|_2
\]

System Gain

\[
G = U \Sigma V^H
\]

\( U, V \) : Unitary Matrices

\( \sigma_1, \ldots, \sigma_p \) : Singular Values

\[
(\sigma_1 > \sigma_2 > \cdots > \sigma_p)
\]

\[
\sigma_i = \sqrt{\lambda_i(G^H G)} \quad \lambda_i : i\text{-th eigenvalue}
\]

Maximum Singular Value
\[
\tilde{\sigma}(G) = \sigma_1 = \max_{u \neq 0} \frac{\|y\|_2}{\|u\|_2}
\]

Minimum Singular Value
\[
\sigma(G) = \sigma_p = \min_{u \neq 0} \frac{\|y\|_2}{\|u\|_2}
\]
\( \sigma -\text{plot} \) [SP05, p. 79]

SISO: Absolute value \( |g(j\omega)| \)

MIMO: Maximum singular value \( \bar{\sigma}(G(j\omega)) \)

[Ex.]

\[
G(s) = \begin{bmatrix}
\frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+2} \\
\frac{s+1}{(s+2)(s+3)} & \frac{5(s+1)}{s^2+0.1s+10}
\end{bmatrix}
\]

\[
\begin{array}{ccc}
u(s) & \xrightarrow{G(s)} & y(s) \\
y(\omega) &=& G(j\omega)u(\omega)
\end{array}
\]

\( \sigma -\text{plot of } G \)

Extension of Bode gain plot to MIMO Systems

MATLAB Command

\[
\begin{align*}
G(1,1) &= 10*tf([1 1],[1 0.2 100]) ; \\
G(1,2) &= tf([1],[1 1]) ; \\
G(2,1) &= tf([1 2],[1 0.1 10]) ; \\
G(2,2) &= 5*tf([1 1],[1 5 6]) ; \\
figure \\
sigma(G) ;
\end{align*}
\]
$H_\infty$ Norm [SP05] (p. 158)

System Gain  $\|G(s)\|_\infty = \max_\omega \sigma(G(j\omega))$

$G(s)$: Stable system

[Ex.]

$G(s) = \begin{bmatrix}
\frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+2}
\frac{s+1}{s^2+0.1s+10} & \frac{1}{5(s+1)}
\frac{(s+2)(s+3)}{}
\end{bmatrix}$

$\|G(s)\|_\infty = 50.25$ (34.02 [dB])

G. H. Hardy

MATLAB Command

$hinfG = \text{normhinf}(G)$
[Ex.] Spinning Satellite: $\sigma$-plot

Plant: $P(j\omega) = \begin{bmatrix} \frac{j\omega-100}{100-\omega^2} & \frac{10j\omega+10}{100-\omega^2} \\ -\frac{10j\omega-10}{100-\omega^2} & \frac{j\omega-100}{100-\omega^2} \end{bmatrix}$

$a = 10\text{ rad/s}$

$\sigma(P(j\omega))$

$\omega = 0.01[\text{rad/s}]$   0.1   1   10   100
$\bar{\sigma} = 0.05[\text{dB}]$   0.13   0.96   $\infty$   $-19.0$
$\sigma = 0.03[\text{dB}]$   $-0.04$   $-0.78$   $-6.0$   $-20.8$

for each fixed $\omega$ $\|P(s)\|_{\infty} = \infty$

MATLAB Command

```
figure
sigma(Pnom)
```
Robust and Optimal Control

Instructor:  Prof. Masayuki Fujita (S5-303B)

Schedule:   7\textsuperscript{th}, 14\textsuperscript{th}, 21\textsuperscript{st}, 28\textsuperscript{th} April, 12\textsuperscript{th}, 19\textsuperscript{th}, 26\textsuperscript{th}, May, 2\textsuperscript{nd}, June

Units:      1 unit

Teaching Assistants (TA):
             Junya Yamauchi, Riku Funada (S5-303A)

Reference:
[SP05] S. Skogestad and I. Postlethwaite,  

[ZD97] K. Zhou and J. C. Doyle,  

Grading: Reports on 4\textsuperscript{th} (40\%), 6\textsuperscript{th} (45\%) and 8\textsuperscript{th} (15\%) classes  
\(\text{MATLAB: Robust Control Toolbox}\)  
(This grading scheme is tentative. To be announced.)
Computer Access:

Install Guide: http://tsubame.gsic.titech.ac.jp/MATLAB-TAH

(Test Commands in Robust Control Toolbox: check pp. 12-13)

Office Hour (Technical Support):

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Course Web: Information on the lecture will be continuously updated

http://www.fl.ctrl.titech.ac.jp/course/ROC/index.html

Tokyo Tech OCW

http://www.fl.ctrl.titech.ac.jp

Robust and Optimal Control, Spring, 2015

Instructor

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George Zames (1934-1997)

G. Zames, IEEE TAC, 26, 1981

Frustration with LQG control ($H_2$ control)

• Formulation of the optimization problem not on time domain but on frequency domain

$H_{\infty}$ control

1939 The World War II occurred.
Escaping to Europe through Lithuania
Witnessed by Soviet’s tank
Through Russia, Siberia and Japan sea,

1941 Arrival in Kobe
Sugihara “Sempo” Chiune, consular officer of Japan, helped him a lot.
Leaving for Canada
Optimization in Feedback Control

“Feedback Performance = Sensitivity”

Sensitivity optimization with $H_\infty$ norm

$$\min_K \| S \|_\infty = \min_K \| (I + PK)^{-1} \|_\infty$$

$$= \min_Q \| I - PQ \|_\infty \quad \text{(Affine in } Q \text{)}$$

(System Gain)

Sensitivity from Reference to Error

$$e = (I + PK)^{-1} r = Sr$$

A.H. Haddad (Ed.), IEEE TAC 1987

Editorial
Why $H$-Infinity?

The title of this editorial came from a paper that was promised to us by one of the Associate Editors at IEEE/AC (AEAC) several years ago. The paper was supposed to provide a critical look at the $H_\infty$ approach to the design of multivariable control systems. We have hopes that the paper will still be written. A domain, why do we have to resort to such a fancy term? We now see other mathematical terms surfacing in other papers such as $L_1$ or $\Omega$, to name a few recent ones. What is wrong with “integral of the absolute value” performance index as a parallel to the well-known quadratic index? My only explanation for the phenomena
Weighted Sensitivity  [SP05, p. 60]

\[ \| S \|_\infty \text{ Small?} \]

\[
z = W_P (I + PK)^{-1} r = W_P S r
\]

\[ \| W_P S \|_\infty \text{ Small!} \]

\[ W_P : \text{Performance weight transfer function matrix} \ [\text{SP05, pp. 62, 80}] \]

\[
W_P(s) = \begin{bmatrix}
w_{p1}(s) & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & w_{pn}(s)
\end{bmatrix}
\begin{bmatrix}
w_p(s) & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & w_p(s)
\end{bmatrix}
\]
Performance Weight $W_P$  [SP05, pp. 62, 80]

First-order Performance Weight

$$w_p(s) = \frac{1}{M_S} s + \omega_b$$

High-order Performance Weight

$$w_p(s) = \frac{(s/M_S^{1/n} + \omega_b)^n}{(s + \omega_b A^{1/n})^n}$$

$A$ : $1/|w_p(j\omega)|$ at low frequencies

$M_S$ : $1/|w_p(j\omega)|$ at high frequencies

$\omega_b$ : the frequency at which the asymptote of $1/|w_p(j\omega)|$ crosses 1, and the bandwidth requirement approximately
Nominal Performance (NP) \[\text{[SP05, p. 81]}\]

Given a controller \( K(s) \),

\[
\| W_P(s) S(s) \|_\infty < 1
\]

\( \bar{\sigma}(S(j\omega)) < \frac{1}{\bar{\sigma}(W_P(j\omega))} \quad \forall \omega \)

\( \bar{\sigma}(W_P(j\omega)S(j\omega)) < 1 \quad \forall \omega \)

Nominal Performance (NP) Test

A set of \( S \)s

A class of \( S \)s

\[
\begin{cases}
\sigma(A)\bar{\sigma}(B) \leq \bar{\sigma}(AB) \\
\bar{\sigma}(AB) \leq \bar{\sigma}(A)\bar{\sigma}(B) \\
\bar{\sigma}(A^{-1}) = \frac{1}{\sigma(A)} \\
\|A\|_\infty = \bar{\sigma}(A)
\end{cases}
\]
Nominal Performance Test in SISO Systems [SP05, p. 60]

\[(NP) \quad |S(j\omega)| < \frac{1}{|\omega_p(j\omega)|} \quad \forall \omega\]

[Ex.] \[S = \frac{s^2 + s}{s^2 + 0.7s + 0.07} \]

\[\omega_p(s) = \frac{1}{M_S} s + \omega_b \]

1) \(\omega_{P1} \quad M_S = 2, \quad \omega_b = 0.01\)

(NP) \(\bigcirc\)

\(M_S : \text{small} \quad \omega_b : \text{fast}\)

2) \(\omega_{P2} \quad M_S = 1.2, \quad \omega_b = 0.06\)

(NP) \(\times\)

\(M_S : \text{large}\)

3) \(\omega_{P3} \quad M_S = 1.7, \quad \omega_b = 0.06\)

(NP) \(\bigcirc\)
Nominal Performance Test in SISO Systems [SP05, p. 60]

\[
\|W_P(s)S(s)\|_\infty < 1
\]

**[Ex.]**

\[
S = \frac{s^2 + s}{s^2 + 0.7s + 0.07}
\]

\[
w_p(s) = \frac{1}{M_S} s + \omega_b
\]

1) \(w_{P1} M_S = 2, \quad \omega_b = 0.01\)

\[
\left(\text{NP}\right) \quad \bigcirc
\]

\(M_S : \text{small} \quad \omega_b : \text{fast}\)

2) \(w_{P2} M_S = 1.2, \quad \omega_b = 0.06\)

\[
\left(\text{NP}\right) \quad \times
\]

\(M_S : \text{large}\)

3) \(w_{P3} M_S = 1.7, \quad \omega_b = 0.06\)

\[
\left(\text{NP}\right) \quad \bigcirc
\]
Spinning Satellite: Performance Weight

Performance Weight \( W_P \)

\[
W_P(s) = \begin{bmatrix} w_p(s) & 0 \\ 0 & w_p(s) \end{bmatrix} = \begin{bmatrix} w_{p1} & 0 \\ 0 & w_{p2} \end{bmatrix}
\]

Specifications \( w_p(s) = \frac{1}{M_S} s + \omega_b \)

- \( M_S \leq 2 \rightarrow M_S = 2 \)
- the steady state error \( e \leq 0.01 \rightarrow A = 0.01 \)
- Poles on the imaginary axis
  \( p = \pm a_j = \pm 10 \ j \)

Gain crossover frequency

\( \omega_c > 1.15 |p| = 11.5 \ \text{rad/s} = \omega_b \)

Phase stabilization [SP05, p. 194]

\[ \{ \ \omega_c < \text{System bandwidth of Actuator/Sensor/Controller} \ \} \]

MATLAB Command

\[
\begin{align*}
W_P &= \text{tf}([1/M_S \ \omega_b], [1 \ \omega_b \ A]); \\
WP &= \text{eye}(2)*W_P; \\
\text{figure} \\
\text{sigma}(WP) \\
\text{hold on; grid on;}
\end{align*}
\]
Spinning Satellite: Nominal Performance

Plant

\[ P(s) = \begin{bmatrix}
\frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\
\frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100}
\end{bmatrix} \]

Target Loop Transfer Function

\[ L(s) = PK_I = \frac{900}{s(s + 30)}I_2 \]

Controller: Inverse-based Controller \( K_I \)

\[ K_I(s) = P^{-1}(s) \begin{bmatrix}
\frac{900}{s(s+30)} & 0 \\
0 & \frac{900}{s(s+30)}
\end{bmatrix} \]

(Output) Sensitivity Function

\[ S(s) = (I + PK_I)^{-1} \]

\[ \|WP_S\|_\infty = 0.8935 < 1 \quad NP \]

MATLAB Command

\[
\begin{align*}
KI &= \text{inv}(P\text{nom})*\text{tf}([1],[1 30 0])*\text{diag}([900 900]); \\
FI &= \text{loopsens}(P\text{nom},KI); \\
\text{sigma}(FI.So) &; \\
\text{hinfSo} &= \text{normhinf}(WP*FI.So)
\end{align*}
\]
Sensitivity Minimization

Optimal Sensitivity Problem

Find a stabilizing controller $K$ which minimizes $\|W_P(s)S(s)\|_\infty$

Intractable

Sensitivity Minimization Problem

Given $\gamma > \gamma_{min}$, find all stabilizing controllers $K$ such that

$\|W_P(s)S(s)\|_\infty < \gamma$ \hspace{1cm} $\gamma$ -iteration

1) $\|W_PS\|_\infty < \gamma_1 \hspace{1cm} \exists K_1$
2) $\|W_PS\|_\infty < \gamma_2 \hspace{1cm} \exists K_2$
3) $\|W_PS\|_\infty < \gamma_3 \hspace{1cm} \times \text{no } K_3$

$\vdots \hspace{1cm} \vdots \hspace{1cm} Q \text{ Parameterization}$

$\frac{1}{\gamma_{opt}}W_P$

$\frac{1}{\bar{\sigma}(1/\gamma_{opt}W_P)}$

$\bar{\sigma}(S')$

$\omega \text{ [rad/s]}$
“Respect the unstable”

Bode lecture, CDC, 1989

Sensitivity for MIMO Systems [SP05, p. 70]

Output Sensitivity Function: $S_o(s) = (I + P(s)K(s))^{-1}$

Input Sensitivity Function: $S_i(s) = (I + K(s)P(s))^{-1}$

For SISO Systems $S_i = S_o$

but for MIMO Systems $PK \neq KP \rightarrow S_i \neq S_o$

Good disturbance rejection at output does not always mean good rejection at input
Standard Feedback Configuration with Weights [SP05, p. 363]

Sensitivity Minimization Problem

\[
\text{find } K(s) \text{ s.t. } \|W_e(s)S_o(s)W_d(s)\|_\infty < \gamma
\]
Generalized Plant  [SP05, pp. 108, 543]

Lower Linear Fractional Transformation (LFT)

Given \( r \), find all stabilizing controllers \( K \) such that

\[
\begin{bmatrix}
\dot{z} \\
y
\end{bmatrix} = G(s) \begin{bmatrix}
w \\
u
\end{bmatrix} = \begin{bmatrix}
G_{11}(s) & G_{12}(s) \\
G_{21}(s) & G_{22}(s)
\end{bmatrix} \begin{bmatrix}
w \\
u
\end{bmatrix}
\]

\( u = K(s)y \)

\( z = F_l(G, K)w \)

\( F_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \)

\( H_\infty \) Sub-optimal Control Problem

Given \( \gamma > \gamma_{min} \), find all stabilizing controllers \( K \) such that

\( \| F_l(G, K) \|_\infty < \gamma \)
1. Nominal Stability and Nominal Performance

1.1 Nominal Stability [SP05, Sec. 4.1, 4.7, 4.8]
1.2 Multivariable Frequency Response Analysis [SP05, Sec. 3.3, A.3, A.5]
1.3 Nominal Performance [SP05, Sec. 2.8]
1.4 Sensitivity Minimization [SP05, Sec. 2.8]

Reference:

2. Robustness and Uncertainty

2.1 Why Robustness? [SP05, Sec. 4.1.1, 7.1, 9.2]

2.2 Representing Uncertainty [SP05, Sec. 7.2, 7.3, 7.4]

2.3 Uncertain Systems [SP05, Sec. 8.1, 8.2, 8.3]

2.4 Systems with Structured Uncertainty [SP05, Sec. 8.2]

Reference:

Poles [SP05, 4.4]

[SP05, Definition 4.6] (p. 135)

The poles \( p_i \) of a system with state-space description \[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\] are the eigenvalues \( \lambda_i(A), \ i = 1, \cdots, n \) of the matrix \( A \). The pole or characteristic polynomial \( \phi(s) \) is defined as

\[
\phi(s) = \det(sI - A) = \prod_{i=1}^{n} (s - p_i)
\]

Thus the poles are the roots of the characteristic equation

\[
\phi(s) = \det(sI - A) = 0
\]

[SP05, Theorem 4.4] (p. 135)

The pole polynomial \( \phi(s) \) corresponding to a minimal realization of a system with transfer function \( G(s) \) is the least common denominator of all non-identically zero minors of all orders of \( G(s) \).

[SP05, Theorem 4.3] (p. 135)

A linear dynamic system \( \dot{x} = Ax + Bu \) is stable if and only if all the poles are in the open left-half plane (LHP); that is

\[
\text{Re}(p_i) = \text{Re}\{\lambda_i(A)\} < 0, \ \forall i
\]

A matrix \( A \) with such a property is said to be “stable” or Hurwitz.
Zeros [SP05, Sec. 4.5]

[SP05, Definition 4.7] (p. 138)

\( z_i \) is a zero of \( G(s) \) if the rank of \( G(z_i) \) less than the nominal rank of \( G(s) \). The zero polynomial is defined as

\[
z(s) = \prod_{i=1}^{n_z} (s - z_i)
\]

where \( n_z \) is the number of finite zeros of \( G(s) \).

[SP05, Theorem 4.5] (p. 139)

The zero polynomial \( z(s) \), corresponding to a minimal realization of the system, is the greatest common divisor of all the numerators of all order-\( r \) minors of \( G(s) \), where \( r \) is the normal rank of \( G(s) \), provided that these minors have been adjusted in such a way as to have the pole polynomial \( \phi(s) \) as their denominator.

[SP05, Ex. 4.10] (pp. 136, 139)

\[
G(s) = \begin{bmatrix}
\frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\
-1 & \frac{1}{s+2} & \frac{1}{s+2}
\end{bmatrix}
\]

Poles \( p = 1, -1, -2, -2 \)  
Zeros \( z = 1 \)
Motivating Example for Internal Stability in SISO Systems

[SP05, Ex. 4.16] (p. 144)

\[ P(s) = \frac{s - 1}{s + 1}, \quad K(s) = \frac{k(s + 1)}{s(s - 1)}, \quad k > 0 \]

**Sensitivity**

\[ S = \frac{s}{s + k} \quad (n \to y) \]

**Load Sensitivity**

\[ PS = \frac{s(s - 1)}{(s + 1)(s + k)} \quad (d \to y) \]

**Noise Sensitivity**

\[ KS = \frac{s + 1}{(s - 1)(s + k)} \quad (n \to u) \]

**Step Response**  \( (k = 2) \)

![Step Response Graphs](attachment://step_response.png)

Unstable

Stable?
Youla Parameterization (Q Parameterization)

Case 1: Stable Plant $P(s)$ [SP05, p. 148]

All Stabilizing Controllers: $K(s) = \frac{Q(s)}{1 - P(s)Q(s)}$

$Q$ -parameter $Q(s)$: Proper Stable Transfer Function

Gang of Four

$$S = \frac{1}{1 + PK} = 1 - PQ$$

$$PS = \frac{P}{1 + PK} = P(1 - PQ)$$

$$T = \frac{PK}{1 + PK} = PQ$$

$$KS = \frac{K}{1 + PK} = Q$$
Youla Parameterization

Case 2: Unstable Plant $P(s)$ [SP05, p. 149]

Coprime Factorization [SP05, p. 122]

$$P(s) = \frac{N(s)}{M(s)}$$

Coprime: No common right-half plane (RHP) zeros

$N(s), M(s)$: Proper Stable Transfer Functions

[SP05, Ex. 4.1] $P(s) = \frac{(s - 1)(s + 2)}{(s - 3)(s + 4)}$

$\rightarrow N(s) = \frac{s - 1}{s + 4}, M(s) = \frac{s - 3}{s + 2}$ (*)

Bezout Identity $NX + MY = 1 \quad \leftrightarrow \quad M(s), N(s)$: Coprime

$X(s), Y(s)$: Proper Stable Transfer Functions

[SP05, Ex.] $M(s), N(s): (*) \quad \rightarrow \quad X(s) = \frac{s + 32}{2s + 4}, Y(s) = \frac{s - 16}{2s + 8}$

[Ex.] $5x + 3y = 1 \quad x, y$: Integer $\quad \rightarrow \quad x = -1 - 3q, y = 2 + 5q$

$q$: Integer
Youla Parameterization

Case 2: **Unstable Plants** \( P(s) \) [SP05, p. 149]

A Stabilizing Controller

\[
K(s) = \frac{X(s)}{Y(s)} \quad (Q(s) = 0)
\]

[SP05, Ex.] \( P(s) = \frac{(s - 1)(s + 2)}{(s - 3)(s + 4)} \), \( X(s) = \frac{s + 32}{2s + 4} \), \( Y(s) = \frac{s - 16}{2s + 8} \)

\[
K(s) = \frac{X(s)}{Y(s)} = \frac{s^2 + 36s + 128}{s^2 - 14s - 32}
\]

All Stabilizing Controllers

\[
K(s) = \frac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)}
\]

\[
\begin{align*}
N &= P, M = 1, X = 0, Y = 1 & K &= \frac{Q}{1 - PQ}
\end{align*}
\]

Gang of Four

\[
\begin{align*}
S &= \frac{M(Y - NQ)}{Y - NQ} & T &= \frac{N(X + MQ)}{X + MQ}
\end{align*}
\]

\[
\begin{align*}
PS &= \frac{N(Y - NQ)}{Y - NQ} & KS &= \frac{M(X + MQ)}{X + MQ}
\end{align*}
\]

Affine Functions of \( Q \)
All Stabilizing Controllers (Computation)

State Space Representation [SP05, p. 124]

\[ P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = M_l^{-1} N_l \]

All Stabilizing Controllers

\[ K(s) = (Y_r - QN_l)^{-1}(X_r + QM_l) \]

Let matrices \( F, H \) be such that \( A + BF, A + HC \) are stable

\[
\begin{bmatrix} X_r & Y_r \\ N_l & M_l \end{bmatrix} = \begin{bmatrix} A + HC & -B - HD & H \\ F & I & 0 \\ C & -D & I \end{bmatrix}
\]

System Structure on Controllers

If \( Q = 0 \), then \( K \) is

State Feedback + Observer

\[
\begin{cases} u = F \hat{x} \\ \dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x}) \end{cases}
\]
Two degrees of freedom Controller [SP05, p. 147]

Parameterize $F(s) = N_l(s)R(s)$ \hspace{1cm} \text{\textbf{R(s) : Stable matrix}}

$$P(s) = M_l^{-1} N_l$$

$$N_l = \begin{bmatrix} A + HC & -B - HD \\ C & -D \end{bmatrix}$$

$$M_l = \begin{bmatrix} A + HC & H \\ C & I \end{bmatrix}$$
Norm [SP05, A.5]

Key properties
1. Non-negative $\|e\| \geq 0$
2. Positive $\|e\| = 0 \iff e = 0$
3. Homogeneous $\|\alpha e\| = |\alpha|\|e\|$, $\forall \alpha : \text{scalar}$
4. Triangle inequality $\|e_1 + e_2\| \geq \|e_1\| + \|e_2\|$

Vector Norm $\|a\|_p = \left(\sum_i |a_i|^p\right)^{1/p}$

[Ex.] $\|a\|_1 = \sum_i |a_i|$, $\|a\|_2 = \sqrt{\sum_i |a_i|^2}$

$\|a\|_\infty = \max_i |a_i|$

Matrix Norms

Induced Matrix Norm $\|A\|_{ip} = \max_{w \neq 0} \frac{\|Aw\|_p}{\|w\|_p}$

[Ex.] $\|A\|_{i2} = \bar{\sigma}(A) = \sqrt{\rho(A^HA)}$
Norm [SP05, A.5]

**Signal Norms** \( \|e(t)\|_p = \left( \int_{-\infty}^{\infty} \sum_i |e_i(\tau)|^p d\tau \right)^{1/p} \)

[Ex.] Integral absolute error

“Energy of signal”

( \( L_2 \)-norm, \( L \): Lebesgue space)

“maximum value over time”

\( \|e(t)\|_\infty = \max_t \left( \max_i |e_i(t)| \right) \)

cf. “average power”

\( \|e(t)\|_{pow} = \sqrt{\frac{1}{2T} \int_{-T}^{T} \sum_i |e_i(t)|^2 dt} \)

**System Norms** \( f(s), G(s) \): Linear, stable, time invariant

**SISO**

\[ f(s) \]

\( \|f(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(j\omega)|^2 d\omega} \)

\( \|f(s)\|_\infty = \max_\omega |f(j\omega)| \)

**MIMO**

\[ G(s) \]

\( \|G(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left( G(j\omega) \bar{H} G(j\omega) \right) d\omega} \)

\( \|G(s)\|_\infty = \max_\omega \bar{\sigma}(G(j\omega)) = \max_{w \neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} \)
**$H_\infty$ Space, $H_2$ Space** [SP05, 4.10]

**$H_\infty$ Space**

(All strictly proper and real rational stable transfer matrices: $R\mathcal{H}_\infty$)

Banach space of matrix or scalar valued functions that are analytic and bounded in the open right-half plane. The corresponding norm is

$$\|G(s)\|_\infty = \max_\omega \bar{\sigma}(G(j\omega))$$

**$H_2$ Space**

(All strictly proper and real rational stable transfer matrices: $R\mathcal{H}_2$)

Hilbert space of matrix or scalar valued functions that are analytic in open right-half plane. The corresponding norm is

$$\|G(s)\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} (G(j\omega)^H G(j\omega)) \, d\omega \right)^{1/2}$$

$$\|G'(s)\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_i \sigma_i^2(G(j\omega)) \, d\omega \right)^{1/2}$$
**$H_\infty$ Space**

**Definition**  A transfer function $G(s)$ is said to be in the space $H_\infty$ if

$$\|G(s)\|_\infty := \sup_{\text{Re}(s)>0} |G(s)| < \infty \quad (\mathcal{H} : \text{Hardy space})$$

Note that if $G(s)$ is in $H_\infty$ then all its poles must be in the left half plane (LHP) and hence this will be a stable transfer function.

**Maximum Modulus Theorem**  If $G(s)$ is in $H_\infty$ then

$$\|G(s)\|_\infty := \sup_{\text{Re}(s)>0} |G(s)| = \sup_{-\infty<\omega<\infty} |G(j\omega)|$$

This result shows that the $H_\infty$-norm can be calculated by just examining $G(s)$ for $s$ on the imaginary axis and it is not required to consider $s$ in the whole of RHP.

**Theorem**  For a stable LTI system with transfer function $G(s)$, its gain is given by

$$\sup_{u \neq 0} \frac{\|y\|_2}{\|u\|_2} = \sup_{\hat{u} \neq 0} \frac{\|\hat{y}\|_2}{\|\hat{u}\|_2} = \|G(s)\|_\infty$$
Difference between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms [SP05, pp. 75, 159]

Minimizing $\mathcal{H}_2$ norm (LQG)

Push down “whole thing”
(all singular values over all frequencies)

Average direction, average frequency

$$u = \delta(t) \xrightarrow{u} \begin{bmatrix} S \end{bmatrix} y \xrightarrow{\text{max}} \max \|y(t)\|_2$$

Minimizing $\mathcal{H}_\infty$ norm

Push down
“peak of maximum singular value”

Worst direction, worst frequency

$$\|u(t)\|_2 = 1 \xrightarrow{u} \begin{bmatrix} S \end{bmatrix} y \xrightarrow{\text{max}} \max \|y(t)\|_2$$

Induced Matrix Norm

$\times$ Induced Matrix Norm

$\times$ Multiplicative property

$$\|A(s)B(s)\|_2$$

$$\leq \|A(s)\|_\infty \|B(s)\|_\infty$$

Induced Matrix Norm

Multiplicative property
**Sensitivity and Feedback Performance**

**Disturbance Attenuation**

**Open-loop**
\[ d_i \rightarrow y \]
\[ y(s) = P(s)d_i(s) \]

**Closed-loop**
\[ y(s) = \frac{1}{1 + P(s)K(s)} P(s)d_i(s) \]

\[ d \rightarrow y \]
\[ y(s) = d(s) \]

\[ y(s) = \frac{1}{1 + P(s)K(s)} d(s) \]

\[ S(s) = \frac{1}{1 + P(s)K(s)} : \text{Sensitivity} \]

\[ |S(j\omega)| \text{ small: good Feedback Performance} \]
Insensitivity to Plant Variations [SP05, p. 23]

\[
G_{yr} = \frac{PK}{1 + PK} \quad \text{(r \to y)}
\]

\[
\frac{dG_{yr}}{G_{yr}} = S \frac{dP}{P}
\]

\[
\left( \frac{dG_{yr}}{dP} = \frac{K}{(1 + PK)^2} = \frac{SPK}{P(1 + PK)} = S \frac{G_{yr}}{P} \right)
\]

\[
G_{ydi} = \frac{P}{1 + PK} \quad \text{(d_i \to y)}
\]

\[
\frac{dG_{ydi}}{G_{ydi}} = S \frac{dP}{P}
\]

\[
\left( \frac{dG_{ydi}}{dP} = \frac{1}{(1 + PK)^2} = \frac{SP}{P(1 + PK)} = S \frac{G_{ydi}}{P} \right)
\]

\[
|S(j\omega)| \text{ small : good Feedback Performance}
\]
Benefits of Feedback

- Disturbance Attenuation
- Insensitivity to Plant Variations
- Stabilization (Unstable Plant)
- Linearizing Effects
- Reference Tracking

\[ G_{er} = \frac{1}{1 + PK} = S \]

\[ |S(j\omega)|: \text{small} \]

\[ \text{Two-degrees-of-freedom Control} \quad \text{Feedback + Feedforward} \]
There exists a frequency range over which the magnitude of the sensitivity function exceeds 1 if it is to be kept below 1 at the other frequency range.

\[ \int_0^\infty \log |S(j\omega)| \, d\omega = 0 \]

\[ |S| < 1 \quad (\log |S| < 0) \]

\[ |S| > 1 \quad (\log |S| > 0) \]

\[ P(s) = \frac{2 - s}{2 + s}, \quad K(s) = \frac{k}{s} \]

\[ S(s) = \frac{1}{1 + P(s)K(s)} \]

[SP05, Ex., p. 170]
Nominal Performance in SISO Systems

\[ |w_P S| < 1 \quad \forall \omega \quad \Leftrightarrow \quad |w_P| < \left| \frac{1}{1 + L} \right| \quad \forall \omega \]

\[ S = \frac{1}{1 + PK} = \frac{1}{1 + L} \]

Nyquist Plot [SP05, p. 281]

- \( |w_P| < |1 + L| \quad \forall \omega \)

- \( |w_P| > |1 + L| \quad \exists \omega \)

\( L \) should be away from \((-1, 0)\) by \(|w_P|\)